1. [15 points] A damped oscillator consists of a spring (with a spring constant $k = 6.00$ N/m), a block of mass $m = 1.50$ kg, and a damping force given by $-b\dot{x}$ (where $b = 0.60$ kg/s). Suppose that the block is initially pulled so that the spring is extended a distance of $x = 0.12$ m and then released from rest.

(a) [3 points] What is the equation of motion for the block?

$$\ddot{x} + \frac{2}{3} \ddot{x} + \omega_0^2 x = 0$$

(b) [3 points] What is the solution to the equation of motion i.e. what is $x(t)$?

$$x = A e^{-\beta t} \cos \omega_1 t \quad \text{with} \quad \omega_1 = \sqrt{\omega_0^2 - \beta^2} = 1.99 \text{ s}^{-1}$$

(c) [3 points] Calculate the time required for the amplitude of the oscillations to fall to one-fifth of its initial value.

$$\frac{A}{5} = A e^{-\beta t} \Rightarrow \frac{1}{5} = e^{-0.2t}$$

$$\ln \frac{1}{5} = -1.609 = -0.2t \quad \Rightarrow \quad t = \frac{1.609}{0.2} = 8.05 \text{ sec}$$

(d) [3 points] How many oscillations are made by the block in this time?

$$T = \frac{2\pi}{\omega_1} = \frac{2\pi}{1.99} = 3.16 \text{ sec}$$

$$\text{# oscillations in } 8.05 \text{ sec} = \frac{8.05}{3.16} = 2.55$$

(e) [3 points] Sketch (roughly) the phase path ($\dot{x}$ vs. $x$).
2. [10 points] A sphere of radius \( R = 15.0 \) m has a density that varies as \( \rho(r) = \rho_0 / r \) and \( \rho_0 = 3.0 \times 10^3 \text{ kg/m}^3 \).

(a) [4 points] What is the total mass of the sphere?

\[
M = \int 4\pi \rho_0 \frac{r^2}{r} dr = 4\pi \rho_0 \int r^2 dr = 4\pi \rho_0 \left[ \frac{r^4}{4} \right]_0^R = \pi \rho_0 R^4 = 3.18 \times 10^7 \text{ kg}
\]

(b) [6 points] What is the gravitational force on an object of mass \( m = 2.0 \) kg situated at a radius \( r = 10.0 \) m, i.e. 5.0 m beneath the outer surface of the sphere? (You can use Gauss’ Law or the Shell Theorem.)

\[
G' = \frac{GmM'}{r^2} = -\frac{GmM}{r^2}
\]

\[
F = \frac{GmM'}{r^2} = 8.38 \times 10^6 \text{ N}
\]

3. [12 points] Spaceships of equal mass \( m \) are in orbits around a planet of mass \( M \) (with \( M \gg m \)). One spaceship is in a circular orbit (of radius \( R \)) and one in an elliptical orbit. The orbits intersect at point A. (Note: you have been given \( G, m, M \) and \( R \). Please express your answers in terms of these variables.)

In circular orbit

\[
U = -\frac{GmM}{R}
\]

(a) [2 points] Which spaceship has the larger total energy?

\[
E_{\text{ellipse}} = -\frac{GmM}{2a} - \frac{GmM}{3R} \Rightarrow \text{Less negative so } E_{\text{ellipse}} > E_{\text{circ}}
\]

(b) [2 points] What is the velocity of the spaceship in the circular orbit?

\[
\frac{1}{2} m V^2 = \frac{1}{2} \frac{GmM}{R} \Rightarrow V = \sqrt{\frac{Gm}{R}}
\]

(c) [2 points] What is the velocity of the spaceship in the elliptical orbit at point A?

\[
E_{\text{ellipse}} = K_A + U_A \Rightarrow K_A = \frac{1}{2} m V_A^2 = -\frac{GmM}{3R} + \frac{GmM}{R} = \frac{2GmM}{3R}
\]

\[
V_A = \sqrt{\frac{4Gm}{3R}}
\]

(d) [2 points] What is the velocity of the spaceship in the elliptical orbit at point B?

\[
E_{\text{ellipse}} = K_B + U_B \Rightarrow K_B = \frac{1}{2} m V_B^2 = -\frac{GmM}{3R} + \frac{GmM}{2R} = \frac{GmM}{6R}
\]

\[
V_B = \sqrt{\frac{GmM}{3R}}
\]

(e) [4 points] What is the eccentricity of the ellipse?

\[
r_{\text{min}} = R = \frac{a}{1+e} \Rightarrow \frac{r_{\text{max}}}{r_{\text{min}}} = \frac{2R}{R} = \frac{1+e}{1-e}
\]

\[
2(1-e) = 1+e \Rightarrow 3e = 1 \Rightarrow e = \frac{1}{3}
\]
4. [13 points] A particle of mass \( m_1 = 8.0 \) kg traveling at an initial velocity \( u_1 = 15.0 \) m/s in the positive x direction collides with a stationary particle of mass \( m_2 = 12.0 \) kg. After the collision, the first particle is traveling with speed \( v_1 = 8.0 \) m/s at an angle \( \psi = 60.0^\circ \) with respect to the x direction.

**Conservation of mom. in x direction:**
\[ m_1 u_1 + 0 = m_1 v_1 \cos \psi + m_2 v_2 \cos \theta \]
\[ \Rightarrow m_2 v_2 \cos \theta = 8.15 - 8.0 \cos 60^\circ = 8.80 \]

**Conservation of mom. in y direction:**
\[ 0 = m_1 u_1 \sin \psi - m_2 v_2 \sin \theta \]
\[ \Rightarrow m_2 v_2 \sin \theta = 8.8 \sin 60^\circ = 55.4 \]

Dividing \( \Rightarrow \frac{\sin \theta}{\cos \theta} = \tan \theta = 0.630 \Rightarrow \theta = 32.2^\circ \)

(a) [4 points] What are the final speed and direction of the second particle? (i.e. find \( v_2 \) and \( \xi \).

**Warning:** do not assume that the collision is elastic; it is not.

(b) [2 points] How much kinetic energy is lost in the collision?

\[ K_i = \frac{1}{2} \cdot 8 \cdot 15^2 = 900 \text{ J} \]

\[ K_f = \frac{1}{2} \cdot 8 \cdot 8^2 + \frac{1}{2} \cdot 12 \cdot 8.67^2 = 256 + 451 = 707 \text{ J} \]

\[ \Rightarrow \Delta KE_{lost} = 900 - 707 = 193 \text{ J} \]

(c) [2 points] What is the velocity of the center of mass (CM) frame? Draw a picture of the situation in the CM frame before the collision, showing clearly the initial velocities \( u_1 \) and \( u'_2 \).

\[ V_{cm} = \frac{m_1 u_1}{m_1 + m_2} = \frac{8.15}{8 + 12} = 0.6 \text{ m/s} \]

\[ u_1' = u_1 - V = 9 \text{ m/s} \]

\[ u_2' = 0 - 6 \text{ m/s} \]

(d) [5 points] Draw a picture of the situation in the CM frame after the collision. What are the final speeds \( v_1' \) and \( v_2' \) of the two particles in the CM frame? What is the scattering angle \( \theta \) in the CM frame?

\[ v_{1y}' = v_{1y} = v_1 \sin \psi = 8.5 \sin 60^\circ = 6.93 \text{ m/s} \]

\[ v_{1x}' = v_{1x} - V = 8 \cos 60^\circ - 6 = 4 - 6 -2 \]

\[ \Rightarrow v_1' = \sqrt{6.93^2 + (-2)^2} = 7.21 \text{ m/s} \]

\[ \tan \theta = \frac{6.93}{2} \Rightarrow \theta = 74^\circ \]

\[ V_{2y}' = V_{2y} = V_2 \sin \theta = 8.67 \sin 32.2 = 4.62 \text{ m/s} \]

\[ V_{2x}' = V_{2x} - V = 8.67 \cos 32.2 - 6 = 1.34 \text{ m/s} \]

\[ \Rightarrow v_2' = \sqrt{4.62^2 + 1.34^2} = 4.81 \text{ m/s} \]

\[ \tan \theta = \frac{4.62}{1.34} = 74^\circ \Rightarrow \theta + \theta = 106 + 74 = 180^\circ \]
PHY 321 Formulae and Constants

Oscillations: \( x + \alpha^2 x = 0 \) \( x(t) = \text{A} \sin(\omega_0 t - \delta) \)

Damped SHM \( x + 2\beta \dot{x} + \alpha^2 x = 0 \) The general solution is:
\[ x(t) = e^{-\beta t}[A_1 \exp(\sqrt{\beta^2 - \alpha^2} t) + A_2 \exp(-\sqrt{\beta^2 - \alpha^2} t)] \]
depending on the relative value of \( \alpha^2 \) and \( \beta^2 \).

For underdamped \( \omega_i = \omega_0 - \beta^2 \)

Gravity: \( \ddot{r} = -G \frac{Mm}{r^2} \dot{r} \) \( \ddot{\theta} = -G \frac{Mm}{r} \) \( \ddot{\phi} = -G \frac{Mm}{r} \phi \)

Gauss' Law: \( \nabla \cdot \Phi = -4\pi G \rho \)

Calculus in spherical coordinates: \( \nabla = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \)

Central force motion: reduced mass \( \mu = \frac{Mm}{M + m} \)

\( L = r \times p \quad l = |L| = \mu r^2 \dot{\theta} \quad K = \frac{1}{2} \mu r^2 (r^2 + 2 \dot{r} \dot{r} \dot{\theta}) = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu \dot{\theta}^2 \)

\( E = K + U = \frac{1}{2} mv^2 + U(r) = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu \dot{\theta}^2 + U(r) = \frac{1}{2} \mu \dot{r}^2 + \phi(r) \)

Orbits in gravitational field (Kepler’s problem): \( U(r) = -G \frac{Mm}{r} = -\frac{k}{r} \)

\( \frac{\alpha}{r} = 1 + \varepsilon \cos \theta \quad \alpha = \frac{j^2}{\mu k} \quad \varepsilon = \sqrt{1 + \frac{2Ej^2}{\mu k^2}} \quad a = \frac{k}{2|E|} = \frac{\alpha}{1 - \varepsilon^2} \)

\( \varepsilon = 0: \text{circle} \quad 0 < \varepsilon < 1: \text{ellipse} \quad \varepsilon = 1: \text{parabola} \quad \varepsilon > 1: \text{hyperbola} \)

Elliptical orbits: \( K > -\frac{1}{2} < U > E = -\frac{k}{2a} \) Kepler’s 3rd Law: \( \tau^2 = 4\pi^2 \frac{\mu}{k} a^3 \)

Astronomical Data: \( G = 6.67 \times 10^{-11} \text{Nm}^2/\text{kg}^2 \)

\( L = T - U \quad \frac{dL}{dt} - \frac{d}{dt} \left( \frac{dL}{d\dot{z}} \right) = 0 \)