Name (please print): ______________________ **PID:** __________

PHY321 **Practice Final Exam** **Spring 2008**

There are **four** questions. Total points = 50. Show all of your work!

1. [15 points] A damped oscillator consists of a spring (with a spring constant k = 6.00 N/m), a block of mass m = 1.50 kg, and a damping force given by $-b\dot{x}$ (where b = 0.60 kg/s). Suppose that the block is initially pulled so that the spring is extended a distance of x = 0.12 m and then released from rest.

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{6.0}{1.5}} = 2.0\ s^{-1}$$

$$\beta = \frac{b}{2m} = \frac{0.6}{2\cdot 1.5} = 0.2\ s^{-1}$$

$$\omega_0 > \beta \Rightarrow \text{underdamped}$$

(a) [3 points] What is the equation of motion for the block?

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$\ddot{x} + 2(0.2)\dot{x} + 2^2 x = 0$$

$$\Rightarrow \boxed{\ddot{x} + 0.4\dot{x} + 4x = 0}$$

(b) [3 points] What is the solution to the equation of motion i.e. what is x(t)?

$$x = Ae^{-\beta t}\cos\omega_1 t \quad \text{with } \omega_1 = \sqrt{\omega_0^2 - \beta^2} = 1.99\ s^{-1}$$

At $t = 0$ $x = A = 0.12\ m$

$$\Rightarrow \boxed{x(t) = 0.12\,e^{-0.2t}\cos(1.99t)}$$

(c) [3 points] Calculate the time required for the amplitude of the oscillations to fall to one-fifth of its initial value.

$$A/5 = Ae^{-\beta t} \Rightarrow \frac{1}{5} = e^{-0.2t}$$

$$\therefore \ln\frac{1}{5} = -1.609 = -0.2t \qquad \therefore t = \frac{1.609}{0.2}$$

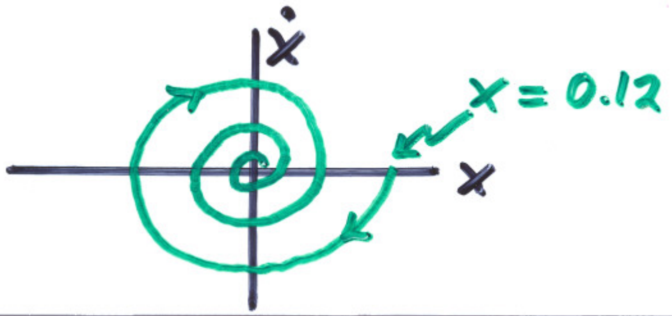
$$\Rightarrow \boxed{t = 8.05\ \text{sec}}$$

(d) [3 points] How many oscillations are made by the block in this time?

$$T = \frac{2\pi}{\omega_1} = \frac{2\pi}{1.99} = 3.16\ \text{sec}$$

$$\#\ \text{oscillations in } 8.05\ \text{sec} = \frac{8.05}{3.16} = \boxed{2.55}$$

(e) [3 points] Sketch (roughly) the phase path ($\dot{x}$ vs. x).



$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad dV = 4\pi r^2 dr$$

2. [10 points] A sphere of radius R = 15.0 m has a density that varies as $\rho(r) = \rho_o r/R$ and ρ_o = 3.0E3 kg/m^3.

$$dM = \rho\, dV = 4\pi \rho r^2 dr$$

(a) [4 points] What is the total mass of the sphere?

$$\Rightarrow M = \int_0^R 4\pi \rho_o \frac{r}{R} r^2 dr = \frac{4\pi \rho_o}{R}\int_0^R r^3 dr$$

$$\therefore M = \frac{4\pi\rho_o}{R}\cdot \frac{1}{4} r^4 \Big|_0^R = \frac{\pi\rho_o}{R}\cdot R^4 = \pi\rho_o R^3 = \boxed{3.18E7 \text{ kg}}$$

(b) [6 points] What is the gravitational force on an object of mass m = 2.0 kg situated at a radius of r = 10.0 m, i.e. 5.0 m beneath the outer surface of the sphere? (You can use Gauss' Law or the Shell Theorem.)

Shell Theorem ⇒

$$M' = \frac{\pi\rho_o}{R}\cdot 10^4 = 6.28E6 \text{ kg}$$

Acts as if all at center

$$\Rightarrow F = \frac{GmM'}{10^2} = \boxed{8.38E\text{-}6\text{N}}$$

G's Law ⇒ $4\pi 10^2 \vec{g} = -4\pi G M'$

$$\therefore \vec{g} = -\frac{GM'}{10^2}$$

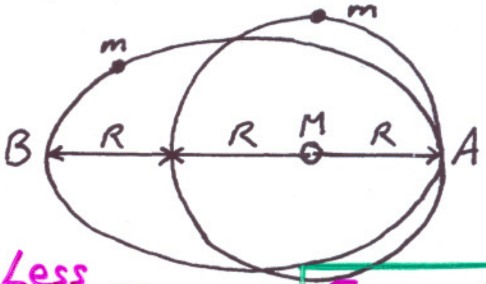
$$\Rightarrow F = mg = -\frac{GmM'}{10^2}$$

3. [12 points] Spaceships of equal mass m are in orbits around a planet of mass M (with M >> m). One spaceship is in a circular orbit (of radius R) and one in an elliptical orbit. The orbits intersect at point A. (Note: you have been given G, m, M and R. Please express your answers in terms of these variables.)

In circular orbit

$$U = -\frac{GmM}{R} \qquad KE = +\frac{1}{2}\frac{GmM}{R}$$

$$\Rightarrow E_{circ.} = -\frac{1}{2}\frac{GmM}{R}$$



(a) [2 points] Which spaceship has the larger total energy?

$$E_{ellipse} = -\frac{GmM}{2a} = -\frac{GmM}{3R} \Rightarrow$$ Less negative so $\boxed{E_{ellipse} > E_{circ}}$

(b) [2 points] What is the velocity of the spaceship in the circular orbit?

$$\frac{1}{2}mv^2 = \frac{1}{2}\frac{GmM}{R} \Rightarrow \boxed{v = \sqrt{\frac{GM}{R}}}$$

(c) [2 points] What is the velocity of the spaceship in the elliptical orbit at point A?

$$E_{ellipse} = K_A + U_A \Rightarrow K_A = \frac{1}{2}mV_A^2 = -\frac{GmM}{3R} + \frac{GmM}{R} = \frac{2}{3}\frac{GmM}{R}$$

$$\Rightarrow \boxed{V_A = \sqrt{\frac{4GM}{3R}}}$$

(d) [2 points] What is the velocity of the spaceship in the elliptical orbit at point B?

$$E_{ellipse} = K_B + U_B \Rightarrow K_B = \frac{1}{2}mV_B^2 = -\frac{GmM}{3R} + \frac{GmM}{2R} = \frac{1}{6}\frac{GmM}{R}$$

$$\therefore \boxed{V_B = \sqrt{\frac{GM}{3R}}}$$

(e) [4 points] What is the eccentricity of the ellipse?

$$r_{min} = R = \frac{\alpha}{1+\varepsilon}$$

$$r_{max} = 2R = \frac{\alpha}{1-\varepsilon}$$

$$\frac{r_{max}}{r_{min}} = \frac{2R}{R} = \frac{1+\varepsilon}{1-\varepsilon}$$

$$\Rightarrow 2(1-\varepsilon) = 1+\varepsilon \quad \therefore 3\varepsilon = 1 \quad \therefore \boxed{\varepsilon = \frac{1}{3}}$$

conservation of mom. in x direction : $m_1 u_1 + 0 = m_1 v_1 \cos\psi + m_2 v_2 \cos\xi$

$\Rightarrow m_2 v_2 \cos\xi = 8 \cdot 15 - 8 \cdot 8 \cdot \cos 60^\circ = 88.0$

4. [13 points] A particle of mass $m_1 = 8.0$ kg traveling at an initial velocity $u_1 = 15.0$ m/s in the positive x direction collides with a stationary particle of mass $m_2 = 12.0$ kg. After the collision, the first particle is traveling with speed $v_1 = 8.0$ m/s at an angle $\psi = 60.0^\circ$ with respect to the x direction.

conservation of mom. in y direction : $0 = m_1 v_1 \sin\psi - m_2 v_2 \sin\xi$

(a) [4 points] What are the final speed and direction of the second particle? (i.e. find v_2 and ξ.) Warning: do not assume that the collision is elastic; it is not.)

$\Rightarrow m_2 v_2 \sin\xi = 8 \cdot 8 \cdot \sin 60^\circ = 55.4$

Dividing $\Rightarrow \frac{\sin\xi}{\cos\xi} = \tan\xi = \frac{55.4}{88.0} = 0.630 \Rightarrow \boxed{\xi = 32.2^\circ}$

substituting $\Rightarrow v_2 = \frac{88.0}{12 \cdot \cos 32.2} = \boxed{8.67 \text{ m/s}}$

(b) [2 points] How much kinetic energy is lost in the collision?

$K_i = \frac{1}{2} \cdot 8 \cdot 15^2 = 900$ J

$K_f = \frac{1}{2} \cdot 8 \cdot 8^2 + \frac{1}{2} \cdot 12 \cdot 8.67^2 = 256 + 451 = 707$

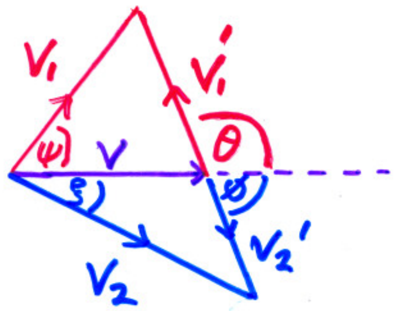
$\Rightarrow KE_{lost} = 900 - 707 = \boxed{193 \text{ J}}$

(c) [2 points] What is the velocity of the center of mass (CM) frame? Draw a picture of the situation in the CM frame before the collision, showing clearly the initial velocities u'_1 and u'_2.

$V_{cm} = \frac{m_1 u_1}{m_1 + m_2} = \frac{8 \cdot 15}{8 + 12} = \boxed{6 \text{ m/s}}$

$u_1' = u_1 - V = 9$ m/s -6 m/s $u_2' = 0 - 6$ m/s

(d) [5 points] Draw a picture of the situation in the CM frame after the collision. What are the final speeds v'_1 and v'_2 of the two particles in the CM frame? What is the scattering angle θ in the CM frame?



$V_{1y}' = V_{1y} = V_1 \sin\psi = 8 \cdot \sin 60 = 6.93$

$V_{1x}' = V_{1x} - V = 8 \cdot \cos 60 - 6 = 4 - 6 = -2$

$\Rightarrow V_1' = \sqrt{6.93^2 + (-2)^2} = \boxed{7.21 \text{ m/s}}$

$\tan\theta = \frac{6.93}{-2} \Rightarrow \boxed{\theta = 106^\circ}$

$V_{2y}' = V_{2y} = V_2 \sin\xi = 8.67 \sin 32.2 = 4.62$ m/s

$V_{2x}' = V_{2x} - V = 8.67 \cos 32.2 - 6 = 1.34$ m/s

$\Rightarrow V_2' = \sqrt{4.62^2 + 1.34^2} = 4.81$ m/s

$\tan\phi = \frac{4.62}{1.34} = 74^\circ \Rightarrow \theta + \phi = 106 + 74 = 180^\circ$

PHY 321 Formulae and Constants

$\omega_0^2 = \frac{k}{m}$ $\quad \beta = \frac{b}{2m}$

Oscillations: SHM $\ddot{x} + \omega_0^2 x = 0$ $\qquad x(t) = A\sin(\omega_0 t - \delta)$

$T = \frac{2\pi}{\omega}$

Damped SHM $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ The general solution is:-
$x(t) = e^{-\beta t}[A_1\exp(\sqrt{(\beta^2 - \omega_0^2)}t) + A_2\exp(-\sqrt{(\beta^2 - \omega_0^2)}t)]$ depending on the relative value of ω_0^2 and β^2.

for underdamped $x(t) = e^{-\beta t}[A\cos\omega_1 t + B\sin\omega_1 t]$ $\quad \omega_1^2 = \omega_0^2 - \beta^2$

Gravity: $\bar{F} = -G\frac{Mm}{r^2}\hat{e}_r$ $\quad U(r) = -G\frac{Mm}{r}$ $\quad \bar{g} = \frac{\bar{F}}{m}$ $\quad \Phi = \frac{U}{m}$ $\quad \bar{g} = -\nabla\Phi$

$V = \frac{4}{3}\pi r^3 \Rightarrow dV = 4\pi r^2 dr$

Gauss' Law: $\nabla \cdot \vec{g} = -4\pi G\rho$ $\qquad$ Know the Shell Theorem!

$\int \vec{g} \cdot d\vec{a} = -4\pi G M$

Calculus in spherical coordinates: $\nabla\psi = \hat{e}_r \frac{\partial\psi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$

$$\nabla \bullet \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Central force motion: $\qquad$ reduced mass $\mu = \frac{Mm}{M+m}$

$$\vec{L} = \vec{r} \times \vec{p} \qquad l = |\vec{L}| = \mu r^2\dot{\theta} \qquad K = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \tfrac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2}$$

$$E = K + U = \tfrac{1}{2}mv^2 + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + V(r)$$

Orbits in gravitational field (Kepler's problem): $\quad U(r) = -G\frac{Mm}{r} = \frac{-k}{r}$

$$\frac{\alpha}{r} = 1 + \varepsilon\cos\theta \qquad \alpha = \frac{l^2}{\mu k} \qquad \varepsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} \qquad a = \frac{k}{2|E|} = \frac{\alpha}{1-\varepsilon^2}$$

$\varepsilon = 0$: circle $\quad 0 < \varepsilon < 1$: ellipse $\quad \varepsilon = 1$: parabola $\quad \varepsilon > 1$: hyperbola

Elliptical orbits: $\langle K \rangle = \frac{-1}{2}\langle U \rangle$ $\qquad E = \frac{-k}{2a}$ $\qquad$ Kepler's 3rd Law: $\tau^2 = 4\pi^2\frac{\mu}{k}a^3$

Astronomical Data: $G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

$L = T - U$ $\qquad \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$