Name (please print): Solutions **PID:** ____________

Midterm Exam #3 **PHY 321 Spring 2008** **April 9, 2008**

Total points = 25. Show all of your work!

1. [8 points] A spherical planet of radius R has a density that is largest in its center, and decreases with distance r from the center as ρ(r) = A(R-r) where A is a constant with the appropriate units.

(a) [4 points] What is the mass of the planet?

$$\text{Volume}, \; V = \frac{4}{3}\pi r^3 \Rightarrow dV = 4\pi r^2 dr$$

$$dM = \rho \, dV \Rightarrow M = \int_0^R 4\pi r^2 \rho \, dr$$

$$M = \int_0^R 4\pi r^2 \cdot A(R-r)\, dr = 4\pi A \int_0^R (Rr^2 - r^3)\, dr$$

$$\therefore M = 4\pi A \left[\frac{1}{3}R^4 - \frac{1}{4}R^4\right] = \frac{4\pi A R^4}{12} = \boxed{\frac{\pi A R^4}{3}}$$

(b) [2 points] Use Gauss' Law to write an expression for the gravitational field g(R) on the surface of the planet.

$$\text{Grav. flux} = \vec{g} \cdot 4\pi R^2 = -4\pi G M$$

$$\Rightarrow \vec{g} = -\frac{GM}{R^2} = \boxed{-\frac{\pi G A R^2}{3} \hat{R}}$$

$$\text{or} \quad \vec{\nabla} \cdot \vec{g} = \frac{1}{r^2} \frac{d}{dr}(r^2 \vec{g}) = -4\pi G \rho = -4\pi G A (R - r)$$

$$\Rightarrow \frac{d}{dr}(r^2 g) = -4\pi G A (Rr^2 - r^3) \Rightarrow r^2 g = -4\pi G A \cdot \left(\frac{1}{3}Rr^3 - \frac{1}{4}r^4\right)$$

(c) [2 points] What is the gravitational potential on the surface (r = R)?

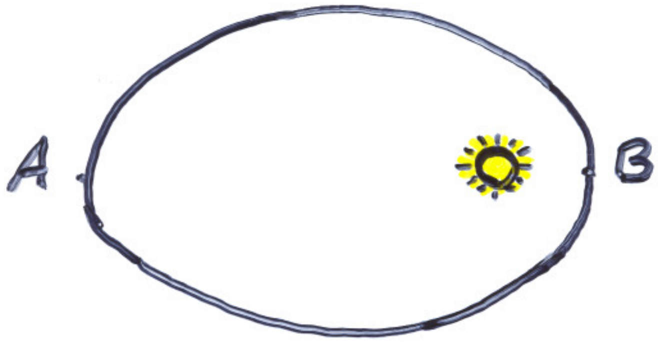
$$\phi = -\frac{GM}{R}$$

$$\Rightarrow \boxed{\phi = -\frac{G\pi A R^3}{3}}$$

$$\therefore R^2 g = -4\pi G A \cdot \frac{R^4}{12}$$

$$\therefore \boxed{\vec{g} = -\frac{\pi G A R^2}{3}}$$

2. [5 points] An asteroid is in an elliptical orbit around a massive star. Compare the following quantities for points A and B on the asteroid's orbit. Answer G, L or E for each question.



(a) E_A <u>E</u> E_B Total Energy remains constant

(b) U_A <u>G</u> U_B $U = -\frac{GMm}{r} \Rightarrow U_A$ is less negative

(c) v_A <u>L</u> v_B If $U_A > U_B \Rightarrow KE_A < KE_B$

(d) ℓ_A <u>E</u> ℓ_B Angular momentum is conserved

(e) $|F|_A$ <u>L</u> $|F|_B$ $\vec{F} \propto \frac{1}{r^2}$

3. [12 points] A satellite (mass = m) is in a circular orbit (of radius R) around the Earth (mass = M).

(a) [4 points] Using the given variables (and G, the gravitational constant) write an expression for the total energy of the satellite.

$$E = U + K = -\frac{GmM}{R} + \frac{1}{2}mv^2$$

Also $\frac{mv^2}{R} = \frac{GmM}{R^2} \Rightarrow v^2 = \frac{GM}{R}$

$$\therefore E = -\frac{GmM}{R} + \frac{1}{2}m\left(\frac{GM}{R}\right) = \boxed{-\frac{1}{2}\frac{GmM}{R}}$$

(b) [4 points] The satellite has a rocket that can fire to instantaneously increase the speed of the satellite. How much additional velocity has to be given to the satellite so that its subsequent motion is unbounded?

Initial velocity, $V_i = V = \sqrt{\frac{GM}{R}}$

We want to add KE so that $E \geq 0$

$$\therefore E = -\frac{GmM}{R} + \frac{1}{2}mV_f^2 = 0$$

$$\Rightarrow V_f^2 = \frac{2GM}{R} \quad \therefore V_f = \sqrt{\frac{2GM}{R}}$$

$\therefore$ Additional velocity $= \Delta V = V_f - V_i$ $\boxed{= (\sqrt{2}-1)\sqrt{\frac{GM}{R}}}$

(c) [4 points] Supposing that this additional velocity was all directed <u>towards</u> the Earth, how close does the satellite get to the Earth?

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{mr^2} + U(r)$$

If $E = 0$ and, at the closest point, $\dot{r} = 0$

$$\Rightarrow U(r) = -\frac{GmM}{r} = \frac{-\ell^2}{2mr^2} \quad \text{with } \ell = mvR$$

$$\therefore r = \frac{v^2R^2}{2GM} = \frac{GM}{R}\cdot\frac{R^2}{2GM} = \boxed{\frac{R}{2}}$$