1. Thornton and Marion, problem 3-20. Do this problem by hand (using algebra), not using a computer. You will need to find the two angular frequencies (call them $\omega_a$ and $\omega_b$) on either side of the resonance where the velocity amplitude has fallen to $v_{max}/\sqrt{2}$. The "full width" of the resonance is then defined as $\omega_b - \omega_a$.

2. Thornton and Marion, problem 3-21. You don't have to use a computer. Just draw four phase paths (one from each quadrant of the xdot vs x phase plot). And calculate the asymptotic path (as $t \to \infty$).

3. Thornton and Marion, problem 3-22.

4. Consider an oscillator consisting of a 0.2 kg mass attached to a spring with a force constant 5 N/m and immersed in a fluid that supplies a damping force represented by $-bv$ with $b = 0.3$ kg/s. What is the nature of the damping (over?, under? or critical?).

Now the oscillator is attached to an external driving force varying harmonically with time as $F_0 \cos \omega t$, where $F_0 = 1.0$ N and $\omega = 4$ rads/s. What is the amplitude of the resulting oscillations?

What is the resonant frequency of the system i.e. what is the frequency $\omega$ that the driving force has to be tuned to in order to produce a maximum amplitude?

What is that amplitude?

What is the Q-value of the system?
For a driven, damped oscillator
\[ x(t) = D \cos(\omega t - \delta) \]
with
\[ D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} \]
\[ \Rightarrow \dot{x}(t) = -\omega D \sin(\omega t - \delta) \]

So the amplitude of the velocity curve is
\[ \dot{x}_{\text{max}} = \frac{\omega A}{\left((\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2\right)^{1/2}} \]

We want to find the value of \( \omega \) for which this is a maximum (resonance) and the value of \( \dot{x}_{\text{max}} \) at resonance.

Set \( \frac{d\dot{x}_{\text{max}}}{d\omega} = 0 \)
After finding $w_{res}$ and the value of $X_{max}$ at $w_{res}$, then find the 2 values of $w$ that correspond to $\frac{1}{\sqrt{2}} X_{max}(w_{res})$

\[ X_{max} \]

\[ X_{max}(w_{res}) \]

\[ \frac{1}{\sqrt{2}} X_{max}(w_{res}) \]

Your job is to find

\[ \Delta w = w_{b} - w_{a} \]
\[
\dot{x}_{\text{max}} = \frac{\omega A}{((\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2)^{1/2}}
\]

\[
\Rightarrow \frac{d\dot{x}_{\text{max}}}{d\omega} = \frac{A}{((\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2)^{1/2}} + \frac{\omega A (\frac{1}{2})(2(\omega_0^2 - \omega^2)(2\omega) + 8\omega \beta^2)}{((\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2)^{3/2}}
\]

Putting over a common denominator \[
A \left[ (\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2 - \omega (-2\omega(\omega_0^2 - \omega^2) + 4\omega \beta^2) \right]
\]
\[
((\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2)^{3/2}
\]
\[
= 0
\]

\[
\Rightarrow (\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2 + 2\omega^2 (\omega_0^2 - \omega^2) - 4\omega^2 \beta^2 = 0
\]

\[
\Rightarrow (\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2 + 2\omega^2) = 0
\]

\[
\Rightarrow (\omega_0^2 - \omega^2)(\omega_0^2 + \omega^2) = 0
\]

\[
\Rightarrow \omega = \omega_0
\]

Velocity Resonance
At resonance the velocity amplitude is

\[ x_{\text{max}} = \frac{w_0 A}{\left((w_0^2 - \omega^2)^2 + 4w_0^2 \beta^2\right)^{1/2}} = \frac{w_0 A}{2w_0 \beta} \]

\[ = \frac{A}{2\beta} \]

We want to find the frequencies where

\[ x_{\text{max}} = \frac{1}{\sqrt{2}} \frac{A}{2\beta} = \frac{w_0 A}{\left((w_0^2 - \omega^2)^2 + 4w_0^2 \beta^2\right)^{1/2}} \]

\[ \Rightarrow (w_0^2 - \omega^2)^2 + 4w_0^2 \beta^2 = 2 \cdot 4 \beta^2 \cdot \omega \]

\[ \Rightarrow (w_0^2 - \omega^2)^2 = 4 \beta^2 \omega^2 \]

\[ \Rightarrow w_0^2 - \omega^2 = \pm 2 \beta \omega \]

\[ \Rightarrow \omega^2 - 2\beta \omega - w_0^2 = 0 \]

\[ \Rightarrow \omega = \pm 2\beta \pm \sqrt{4\beta^2 + 4w_0^2} \]

\[ = \pm \beta \pm \sqrt{w_0^2 + \beta^2} \]

We want positive frequency solutions with \( \omega \) close to \( w_0 \) \( \Rightarrow \)

\[ \omega = \sqrt{w_0^2 + \beta^2} \pm \beta \]
\[ \Rightarrow \omega_a = \sqrt{\omega_0^2 + \beta^2} - \beta \]
\[ \omega_b = \sqrt{\omega_0^2 + \beta^2} + \beta \]
\[ \Delta \omega = \omega_b - \omega_a = 2\beta \]

By definition \[ Q = \frac{\omega_0}{2\beta} \Rightarrow \frac{\omega_0}{2\beta} = 6 \]

\[ \therefore \frac{2}{\beta} = \frac{\omega_0}{6} \Rightarrow \Delta \omega = \frac{\omega_0}{6} \]
For critical damping

\[ x(t) = (A + Bt)e^{-\beta t} \]

\[ \Rightarrow \dot{x}(t) = -\beta A e^{-\beta t} + Be^{-\beta t} - \beta t Be^{-\beta t} \]

\[ = (\beta - \beta (A + Bt)) e^{-\beta t} \]

\[ \therefore \frac{\dot{x}(t)}{x(t)} = \frac{\beta - \beta (A + Bt)}{(A + Bt)} \]

\[ = \frac{\beta}{(A + Bt)} - \beta \]

As \( t \to \infty \),

\[ \frac{\dot{x}(t)}{x(t)} \to -\beta \]
The equation for over-damped oscillations is
\[ x(t) = e^{-\beta t} [A_1 e^{+\omega_2 t} + A_2 e^{-\omega_2 t}] \]
\[ \Rightarrow x(t) = A_1 e^{-(\beta - \omega_2) t} + A_2 e^{-(\beta + \omega_2) t} \]
Substitute \( \beta_1 = \beta - \omega_2 \) and \( \beta_2 = \beta + \omega_2 \)
\[ \Rightarrow x(t) = A_1 e^{-\beta_1 t} + A_2 e^{-\beta_2 t} \]
and \[ x'(t) = -\beta_1 A_1 e^{-\beta_1 t} - \beta_2 A_2 e^{-\beta_2 t} \]

(a) At \( t = 0 \) \[ x_0 = A_1 + A_2 \] \[ ........... (1) \]
\[ v_0 = -\beta_1 A_1 - \beta_2 A_2 \] \[ ........... (2) \]
Multiply (1) by \( \beta_1 \) and add (2) \[ \Rightarrow \beta_1 x_0 + v_0 = \beta_1 A_2 - \beta_2 A_2 \]
\[ \Rightarrow A_2 = \frac{\beta_1 x_0 + v_0}{\beta_1 - \beta_2} \]
or multiply (1) by \( \beta_2 \) and add (2) \[ \Rightarrow \beta_2 x_0 + v_0 = \beta_2 A_1 - \beta_1 A_1 \]
\[ A_1 = \frac{\beta_2 x_0 + V_0}{\beta_2 - \beta_1} \]

(b) If \( A_1 = 0 \) \Rightarrow

\[ x(t) = A_2 e^{-\beta_2 t} \]

and \[ \dot{x}(t) = -\beta_2 A_2 e^{-\beta_2 t} \]

\[ \therefore \frac{\dot{x}(t)}{x(t)} = -\beta_2 \quad \Rightarrow \text{ratio constant for all time} \]

If \( A_1 \neq 0 \) then, after a long time \((t \to \infty)\)

\[ x(t) \to A_1 e^{-\beta_1 t} \]

and \[ \dot{x}(t) \to -\beta_1 A_1 e^{-\beta_1 t} \]

\[ \therefore \frac{\dot{x}(t)}{x(t)} \approx -\beta_1 \quad \text{as } t \to \infty \]
4. Damped oscillations \[ m \ddot{x} = -kx - bx \]

\[ \Rightarrow \ddot{x} + \frac{b}{m} \dot{x} + \frac{k}{m} x = 0 \]

Substitute \( 2\beta = \frac{b}{m} \) and \( \omega_0^2 = \frac{k}{m} \)

\[ \dot{x} + 2\beta \dot{x} + \omega_0^2 x = 0 \]

In this case, \( \beta = \frac{b}{2m} = \frac{0.3}{2 \cdot 0.2} = 0.75 \text{ s}^{-1} \)

\[ \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{0.2}} = 5.0 \text{ s}^{-1} \]

\[ \Rightarrow \omega_0^2 > \beta^2 \Rightarrow \text{ Situation is underdamped} \]

Now we have a damped driven oscillator with \( A = \frac{F_0}{m} = \frac{1.0}{0.2} = 5.0 \text{ m s}^{-2} \) and \( \omega = 4.0 \text{ s}^{-1} \)

The amplitude of the oscillations is

\[ D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2}} = \frac{5}{\sqrt{(5^2 - 4^2)^2 + 4 \cdot 4 \cdot 0.75^2}} \]

\[ \Rightarrow D = \frac{5}{\sqrt{81 + 36}} = \frac{5}{10.8} = 0.462 \text{ m} \]
\[ \omega_{res} = \sqrt{ \omega_0^2 - 2\beta^2 } = \sqrt{5^2 - 2 \cdot 0.75^2} \]
\[ \Rightarrow \omega_{res} = 4.89 \text{ s}^{-1} \]

\[ D_{res} = \frac{5}{\sqrt{(5^2 - 4.89^2) + 4 \cdot 4.89 \cdot 0.75^2}} \]
\[ = \frac{5}{\sqrt{1.27 + 53.7}} = \frac{5}{7.42} \]
\[ \Rightarrow D_{res} = 0.674 \]

\[ Q = \frac{\omega_{res}}{2\beta} = \frac{4.89}{2 \cdot 0.75} = 3.26 \]