1. Consider two quantum systems, labeled 1 and 2. Let $S^{(1)}$ be the Hilbert space of states for system 1, of dimension $N_1$. Let $S^{(2)}$ be the Hilbert space of states for system 2, of dimension $N_2$. We can then describe both quantum systems together by using the tensor product space $S = S^{(1)} \otimes S^{(2)}$, which has dimension $N = N_1 \times N_2$.

2. Let $|n^{(1)}\rangle$ be a complete orthonormal basis for $S^{(1)}$, and $|n^{(2)}\rangle$ be the same for $S^{(2)}$. Then we can write any state of the combined system as a linear combination of tensor products of the basis states:

$$|\Psi\rangle = \sum_{m=1}^{N_1} \sum_{n=1}^{N_2} c_{mn} |m^{(1)}\rangle \otimes |n^{(2)}\rangle.$$ 

We sometimes abbreviate tensor product states as $|m^{(1)}\rangle \otimes |n^{(2)}\rangle = |m^{(1)}|n^{(2)}\rangle = |m,n\rangle$. The simplest kinds of states are called “factorizable,” and can be written as a single tensor product of a state in 1 and a state in 2:

$$|\Psi\rangle = |\varphi^{(1)}\rangle \otimes |\chi^{(2)}\rangle = \left( \sum_{m=1}^{N_1} c_m |m^{(1)}\rangle \right) \otimes \left( \sum_{n=1}^{N_2} d_n |n^{(2)}\rangle \right) = \sum_{m,n} c_m d_n |m^{(1)}\rangle \otimes |n^{(2)}\rangle.$$

Factorizable states are not the most general kind of state of the combined system. (Notice that the most general state can have as many as $N_1 \times N_2$ independent coefficients $c_{mn}$, whereas a factorizable state is determined by the $N_1 + N_2$ coefficients $c_m$ and $d_n$.)

3. The tensor product is easy to understand when systems 1 and 2 represent different quantum systems. What is less obvious, but also true, is that they may represent different degrees of freedom of a single quantum system. Examples include:

   i) the x and y coordinates of a two-dimensional harmonic oscillator
   ii) the spatial and spin degrees of freedom of an electron
   iii) the radial and angular degrees of freedom of an electron

4. Operators defined in $S^{(1)}$ can be extended to $S$ in the obvious way: the operator acts only on the first part of any tensor product state. (If $A$ is the operator in $S^{(1)}$, then define the operator in $S$ as $A^{(1)} \otimes I^{(2)}$.) Similarly, operators defined in $S^{(2)}$ act on the second part of the tensor product state. A familiar example is the angular momentum lowering operator $J_-=J_-^{(1)} + J_-^{(2)}$, which we used to construct the eigenstates of $J^2$ and $J_z$ as linear superpositions of eigenstates of $(J^{(1)})^2$, $J_z^{(1)}$, $(J^{(2)})^2$, and $J_z^{(2)}$. (In this simplest example we are starting at the top of the highest j ladder, with $j=j_1+j_2$, $m=j$, $m_1=j_1$, and $m_2=j_2$.)

$$J_-|j,m\rangle = (J_-^{(1)} + J_-^{(2)}) (|j_1,m_1\rangle \otimes |j_2,m_2\rangle) = (J_-^{(1)}|j_1,m_1\rangle) \otimes |j_2,m_2\rangle + |j_1,m_1\rangle \otimes (J_-^{(2)}|j_2,m_2\rangle)$$

The result is:

$$\hbar \sqrt{j(j+1)-m(m-1)} |j,m-1\rangle = \hbar \sqrt{j_1(j_1+1)-m_1(m_1-1)} |j_1,m_1-1\rangle \otimes |j_2,m_2\rangle$$

$$+ \hbar \sqrt{j_2(j_2+1)-m_2(m_2-1)} |j_1,m_1\rangle \otimes |j_2,m_2-1\rangle$$