Physics 472 – Spring 2008

Homework #11, due Monday, April 21

(Point values are in parentheses.)

1. [5] In class we discussed the non-commutation of rotations in space. For infinitesimal rotations, we used the following identity to derive the commutation relations for angular momentum operators:

$$R(\hat{j},-\delta\alpha')R(\hat{i},\delta\alpha)R(\hat{j},\delta\alpha')R(\hat{i},-\delta\alpha)=R(\hat{k},\delta\alpha\delta\alpha')$$

Prove the above relation. Do this by applying both sides of the equation to an arbitrary vector $\vec{v}$. Keep only terms of order $\delta\alpha$, $\delta\alpha'$, and $\delta\alpha\delta\alpha'$, but ignore terms of order $(\delta\alpha)^2$ or $(\delta\alpha')^2$ (because we know they would cancel if we carried out the calculation consistently to second order). The following vector identity may come in handy: $\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$

2. [7] Griffiths problem 4.56. Remember that you have to write the exponential of an operator as a Taylor series expansion. When you get to part (e), do it first for the special case $\hat{n}=\hat{j}$, i.e. a spin rotation about the y-axis. Apply the spin rotation operator you obtain to the state $|\uparrow\rangle$ (which Griffiths calls χ_+), and check that you get the same answer that we obtained in class long ago for a spinor pointing along an arbitrary direction in the x-z plane. If you are feeling inspired, you can do the general case given in the problem.

3. [2] Suppose there are four variables, *A, A', B, B'* each of which can take one of two values: +1 or –1 (You can think of them as four coins being flipped). There are $2^4 = 16$ possible combinations of outcomes (*A, A', B, B'*) = (1,1,1,1), (1,1,1,–1), (1,1,–1,1), etc. Make a table showing all 16 possibilities. For each combination, calculate the quantity:
F = AB + A'B' + AB' – A'B .
What do you notice?

4. [6] In class we discussed the basis states representing orthogonal linear polarization directions for light. For light polarized along the x or y axes, we called the basis states $|x\rangle$ and $|y\rangle$. We could also choose another direction of linear polarization at an angle θ, with basis states:
$|\theta\rangle=\cos(\theta)|x\rangle+\sin(\theta)|y\rangle$ and $\left|\theta+\frac{\pi}{2}\right\rangle=-\sin(\theta)|x\rangle+\cos(\theta)|y\rangle$. Or we could choose circular polarization, with basis states: $|R\rangle=\frac{1}{\sqrt{2}}(|x\rangle+i|y\rangle)$ and $|L\rangle=\frac{1}{\sqrt{2}}(|x\rangle-i|y\rangle)$.

a) Express the circular polarization basis states, $|R\rangle$ and $|L\rangle$, in terms of $|\theta\rangle$ and $\left|\theta+\frac{\pi}{2}\right\rangle$.

From part (a), you should have obtained:

$$|R\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right) = \frac{e^{i\theta}}{\sqrt{2}}\left(|\theta\rangle + i\left|\theta + \frac{\pi}{2}\right\rangle\right)$$

$$|L\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle\right) = \frac{e^{-i\theta}}{\sqrt{2}}\left(|\theta\rangle - i\left|\theta + \frac{\pi}{2}\right\rangle\right)$$

Now consider a *two-photon* state defined by:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|R\rangle_1|R\rangle_2 + |L\rangle_1|L\rangle_2\right)$$

where the first ket (with subscript 1) refers to photon 1 and the second ket (with subscript 2) refers to photon 2. (Since the photons are traveling in opposite directions, they have zero total angular momentum in this state.)

b) If we use basis states $|\theta\rangle$ and $|\theta + \pi/2\rangle$ for photon 1 and $|\phi\rangle$ and $|\phi + \pi/2\rangle$ for photon 2, show that the state $|\Psi\rangle$ can be written:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\cos(\theta+\phi)\left(|\theta\rangle_1|\phi\rangle_2 - |\theta + \tfrac{\pi}{2}\rangle_1|\phi + \tfrac{\pi}{2}\rangle_2\right) + \sin(\theta+\phi)\left(|\theta\rangle_1|\phi + \tfrac{\pi}{2}\rangle_2 + |\theta + \tfrac{\pi}{2}\rangle_1|\phi\rangle_2\right)\right]$$

c) Suppose that photon 1 enters a measuring apparatus consisting of a linear polarizer with its axis aligned at angle θ followed by an ideal photon detector. If the photon is found to be in state $|\theta\rangle$ it reaches the detector. This result is assigned the value +1. If the photon is found to be in state $|\theta + \pi/2\rangle$ it is blocked by the polarizer and does not reach the detector. This result is assigned the value –1. A similar apparatus measures the polarization state of photon 2, but with the polarizer set at angle ϕ. Find the probability of each of the following results (in terms of θ and ϕ):

P(1,1) = (both photons reach detectors)
P(-1,-1) = (neither photon reaches detector)
P(1,-1) = (photon 1 reaches detector, photon 2 does not)
P(-1,1) = (photon 1 does not reach detector, photon 2 does)

If you are wondering why your answer to part (b) contains the combination $(\theta + \phi)$ rather than $(\theta - \phi)$, it is because we are measuring the two angles with respect to opposite directions of photon propagation. So having the two linear polarizers oriented parallel to each other corresponds to $\theta = -\phi$, or $\theta + \phi = 0$