Physics 472: Summary of Angular Momentum and Spin

Angular Momentum

The two types of angular momentum, orbital angular momentum $\vec{L}$ and intrinsic spin $\vec{S}$, behave nearly the same way. We'll refer to all types of angular momentum as $\vec{J}$. All properties can be derived from the canonical commutation relations:

$$\left[J_x, J_y\right] = i\hbar J_z \text{ and cyclic permutations.}$$

Since $\left[J^2, \vec{J}\right] = 0$, we can find simultaneous eigenstates of $\left[J^2, \vec{J}\right] = 0$ and any component of $\vec{J}$. It is customary to choose the z-component. Then we label our eigenstates by the quantum numbers j and m, where

$$J^2\left|j,m\right\rangle = \hbar^2 j(j+1)\left|j,m\right\rangle$$
$$J_z\left|j,m\right\rangle = \hbar m\left|j,m\right\rangle$$

where j = 0, ½, 1, 3/2, 2 ... and m = -j, -j+1, -j+2, ..., j-1, j

The only difference between $\vec{L}$ and $\vec{S}$ is that *l* takes on only integer values.

The raising and lowering operators for angular momentum are defined as:

$$J_+ = J_x + iJ_y \qquad\qquad J_- = J_x - iJ_y$$

When they act on a state $\left|j,m\right\rangle$, they increase or decrease by 1 the value of *m* without changing the value of *j*:

$$J_+\left|j,m\right\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\left|j,m+1\right\rangle$$
$$J_-\left|j,m\right\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\left|j,m-1\right\rangle$$

Addition of Angular Momentum

If $\vec{J} = \vec{J}_1 + \vec{J}_2$, then the eigenstates of $J^2$ and $J_z$ can be expressed as linear combinations of the tensor product eigenstates of $J_1^2$, $J_{1z}$ and $J_2^2$, $J_{2z}$, using the Clebsch-Gordan coefficients.

$$\left|j,m\right\rangle = \sum_{m_1+m_2=m} C^{j_1 j_2 j}_{m_1 m_2 m}\left|j_1,m_1\right\rangle\left|j_2,m_2\right\rangle$$

where $j = (j_1 + j_2), (j_1 + j_2 - 1), (j_1 + j_2 - 2), \ldots, \left|j_1 - j_2\right|$. You can derive the coefficients by starting at the top state of the top *j* ladder and applying the lowering operator, but you should know how to read the table of Clebsch-Gordan coefficients.

## Spin and Dirac Notation

Because Griffiths uses the spinor notation rather than Dirac notation, this section is intended to clarify the relationship between the two.

If we have a particle with spin s, then the dimension of the Hilbert space associated with the spin degree of freedom is (2s+1). We can work in any orthonormal basis, but we usually choose as our basis states the eigenstates of $S^2$ and $S_z$, labeled $\left|s, m_s\right\rangle$. The eigenvalue equations are:

$$S^2\left|s, m_s\right\rangle = \hbar^2 s(s+1)\left|s, m_s\right\rangle$$
$$S_z\left|s, m_s\right\rangle = \hbar m_s\left|s, m_s\right\rangle$$

If we are dealing with a single particle, then we sometimes omit the "s" in the label, and simply write $\left|m_s\right\rangle$. If s=1/2, we usually substitute $\left|\uparrow\right\rangle$ and $\left|\downarrow\right\rangle$ for $\left|\tfrac{1}{2}\right\rangle$ and $\left|-\tfrac{1}{2}\right\rangle$.

A general spin state $\left|\chi\right\rangle$ can be written as a linear superposition of the basis states:

$$\left|\chi\right\rangle = \sum_{m=-s}^{s} c_m \left|m\right\rangle \qquad \text{where } c_m = \left\langle m \middle| \chi\right\rangle.$$

So we have

$$\left|\chi\right\rangle = \sum_{m=-s}^{s} \left|m\right\rangle\left\langle m \middle| \chi\right\rangle$$

If we remove the ket $\left|\chi\right\rangle$ from both sides, this is just the completeness relation.

Because the Hilbert space is finite, it is sometimes convenient to represent states by column vectors, and operators by matrices. For s=1/2, 1, and 3/2, we get:

$$\left|\chi\right\rangle \to \begin{pmatrix} \left\langle\uparrow\middle|\chi\right\rangle \\ \left\langle\downarrow\middle|\chi\right\rangle \end{pmatrix} \qquad \left|\chi\right\rangle \to \begin{pmatrix} \left\langle 1\middle|\chi\right\rangle \\ \left\langle 0\middle|\chi\right\rangle \\ \left\langle -1\middle|\chi\right\rangle \end{pmatrix} \qquad \left|\chi\right\rangle \to \begin{pmatrix} \left\langle \tfrac{3}{2}\middle|\chi\right\rangle \\ \left\langle \tfrac{1}{2}\middle|\chi\right\rangle \\ \left\langle -\tfrac{1}{2}\middle|\chi\right\rangle \\ \left\langle -\tfrac{3}{2}\middle|\chi\right\rangle \end{pmatrix}$$

For s=1/2, Griffiths uses the spinor notation:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-, \text{ where } \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

You can derive the matrix forms of the spin operators from your knowledge of how the raising and lowering operators act on the spin eigenstates. For spin-1/2, it is customary to express the spin operator matrices in terms of the Pauli spin matrices:

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \quad \text{where} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$