Physics 472: A Brief Introduction to Tensor Product States

1. Consider two quantum systems, labeled 1 and 2. Let $S^{(1)}$ by the Hilbert space of states for system 1, of dimension N_1. Let $S^{(2)}$ by the Hilbert space of states for system 2, of dimension N_2. We can then describe both quantum systems together by using the tensor product space $S = S^{(1)} \otimes S^{(2)}$, which has dimension $N = N_1 \times N_2$.

2. Let $|n\rangle^{(1)}$ be a complete orthonormal basis for $S^{(1)}$, and $|n\rangle^{(2)}$ be the same for $S^{(2)}$. Then we can write any state of the combined system as a linear combination of tensor products of the basis states:

$$|\Psi\rangle = \sum_{m=1}^{N_1}\sum_{n=1}^{N_2} c_{mn}|m\rangle^{(1)} \otimes |n\rangle^{(2)} .$$

We sometimes abbreviate tensor product states as $|m\rangle^{(1)} \otimes |n\rangle^{(2)} = |m\rangle^{(1)}|n\rangle^{(2)} = |m,n\rangle$.

The simplest kinds of states are called "factorizable," and can be written as a single tensor product of a state in 1 and a state in 2:

$$|\Psi\rangle = |\varphi\rangle^{(1)} \otimes |\chi\rangle^{(2)} = \left(\sum_{m=1}^{N_1} c_m |m\rangle^{(1)}\right) \otimes \left(\sum_{n=1}^{N_2} d_n |n\rangle^{(2)}\right) = \sum_{m,n} c_m d_n |m\rangle^{(1)} \otimes |n\rangle^{(2)}$$

Factorizable states are <u>not</u> the most general kind of state of the combined system. (Notice that the most general state can have as many as $N_1 \times N_2$ independent coefficients c_{mn}, whereas a factorizable state is determined by the $N_1 + N_2$ coefficients c_m and d_n.)

3. The tensor product is easy to understand when systems 1 and 2 represent different quantum systems. What is less obvious, but also true, is that they may represent different degrees of freedom of a single quantum system. Examples include:

i) the x and y coordinates of a two-dimensional harmonic oscillator
ii) the spatial and spin degrees of freedom of an electron
iii) the radial and angular degrees of freedom of an electron

4. Operators defined in $S^{(1)}$ can be extended to S in the obvious way: the operator acts only on the first part of any tensor product state. (If A is the operator in $S^{(1)}$, then define the operator in S as $A^{(1)} \otimes I^{(2)}$.) Similarly, operators defined in $S^{(2)}$ act on the second part of the tensor product state. A familiar example is the angular momentum lowering operator $J_- = J_-^{(1)} + J_-^{(2)}$, which we used to construct the eigenstates of J^2 and J_z as linear superpositions of eigenstates of $(J^{(1)})^2$, $J_z^{(1)}$, $(J^{(2)})^2$, and $J_z^{(2)}$. (In this simplest example we are starting at the top of the highest j ladder, with $j=j_1+j_2$, $m=j$, $m_1=j_1$, and $m_2=j_2$.)

$$J_-|j,m\rangle = \left(J_-^{(1)} + J_-^{(2)}\right)\left(|j_1,m_1\rangle \otimes |j_2,m_2\rangle\right) = \left(J_-^{(1)}|j_1,m_1\rangle\right) \otimes |j_2,m_2\rangle + |j_1,m_1\rangle \otimes \left(J_-^{(2)}|j_2,m_2\rangle\right)$$

The result is:

$$\hbar\sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle = \begin{array}{l}\hbar\sqrt{j_1(j_1+1)-m_1(m_1-1)}\,|j_1,m_1-1\rangle \otimes |j_2,m_2\rangle \\ +\hbar\sqrt{j_2(j_2+1)-m_2(m_2-1)}\,|j_1,m_1\rangle \otimes |j_2,m_2-1\rangle\end{array}$$