APPENDIX A: DEALING WITH UNCERTAINTY

1. OVERVIEW

- An uncertainty is always a positive number **$\delta x > 0$**.
- If the uncertainty of **x is 5%,** then **$\delta x = .05x$.**
- If the uncertainty in **x** is **δx**, then the fractional uncertainty in **x** is **$\delta x/x$.**
- If you measure **x** with a device that has a precision of **u**, then **δx** is at least as large as **u**.
- The uncertainty of **$x + y$** or **$x - y$** is **$\delta x + \delta y$**.
- If **d** is data and **e** is expectation:

  The difference is **$\Delta = d - e$**

  % difference is **$100\ (\Delta / e)$**

  They are ***compatible*** IF **$|d - e| < \delta d + \delta e$**
- Fractional uncertainty of **$z = xy$ or x / y** is:

  $$\delta z / z = \delta x / x + \delta y / y$$
- Fractional uncertainty of **$f = x^p\, y^q\, z^r$** is:

  $$\delta f / f = p\delta x / x + q\delta y / y + r\delta z/z$$
- Uncertainty of **$f(x)$** is:

  $$|f(x+\delta x) - f(x)|$$

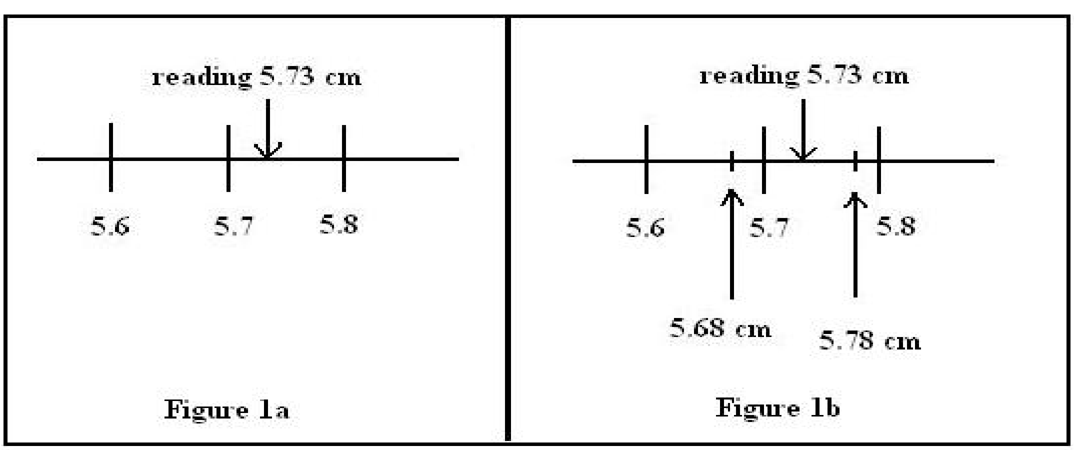
2. ESTIMATING UNCERTAINTIES FOR MEASURED QUANTITIES

(a) Simple Measurements: The smallest division estimate

Suppose we use a meter stick ruled in centimeters and millimeters, and you are asked to measure the length of a rod and obtain the results (see figure 1a)**: L_0 = 5.73 cm.** A good estimate of the uncertainty here is half of the smallest division on the scale, or **0.05** cm. That is, the length of the rod would be specified as:

$$L = 5.73 \pm 0.05\text{cm}$$

This says that you are very confident that the length of the rod falls in the range 5.73 cm – 0.05 cm to 5.73 cm + 0.05 cm, or the length falls in the range of 5.68 cm to 5.78 cm (see figure 1b).



(b) Manufacturer's tolerance

Suppose I purchase a nominally 100 Ω resistor from a manufacturer. It has a gold band on it which signifies a 5% tolerance. What does this mean? The tolerance means $\delta R/R$ = 0.05 = 5%, that is, the fractional uncertainty. Thus, $\delta R = R \times 0.05 = 5\Omega$. We write this as

$$R = R_{nominal} \pm \delta R = 100 \pm 5\ \Omega.$$

It says that the company certifies that the true resistance R lies between 95 and 105Ω. That is, $95 \le R \le 105\Omega$. The company tests all of its resistors and if they fall outside of the tolerance limits the resistors are discarded. If your resistor is measured to be outside of the limits, either (a) the manufacturer made a mistake (b) you made a mistake or (c) the manufacturer shipped the correct value but something happened to the resistor that caused its value to change.

(c) Reading a digital meter.

Suppose I measure the voltage across a resistor using a digital multimeter. The display says 7.45 V and doesn't change as I watch it. The general rule is that the uncertainty is half of the value of the least significant digit. This value is 0.01 V so that half of it is 0.005. Here's why. The meter can only display two digits to the right of the decimal so it must round off additional digits. So if the true value is between 7.445 V and 7.454 V, the display will get rounded to 7.45 V. Thus the average value and its uncertainty can be written as 7.45 ± 0.005V.

When you record this in your notebook, be sure to write 7.45 V. Not 7.450 V. Writing 7.450 V means that the uncertainty is 0.0005 V.

Note that in this example we assumed that the meter reading is <u>steady</u>. If instead, the meter reading is fluctuating, then the situation is different. Now, you need to estimate the range over which the display is fluctuating, then estimate the average value. If the display is fluctuating between 5.4 and 5.8 V, you would record your reading as 5.6 ± 0.2 V. The uncertainty due to the noisy reading is much larger than your ability to read the last digit on the display, so you record the larger error.

3. USING UNCERTAINTIES IN CALCULATIONS

We need to combine uncertainties so that the error bars almost certainly include the true value.

(a) Adding and Subtracting

Let's look at the most basic case. We measure x and y and want to find the error in z.

$$\text{If} \quad z = x + y$$

$$\boxed{\delta z = \delta x + \delta y}$$

$$\text{If} \quad z = x - y$$

$$\boxed{\delta z = \delta x + \delta y}$$

- Note that the uncertainty for <u>subtracting</u> has *exactly* the same form as for <u>adding</u>.
- The most important errors are simply the biggest ones.

Example:

$$(7 \pm 1\ \text{kg}) - (5 \pm 1\ \text{kg}) \quad = \quad 2 \pm 2\ \text{kg}$$

(b) Multiplying and Dividing

If: **a = b x c**, the rule is:

$$\boxed{\frac{\delta a}{a} = \frac{\delta b}{b} + \frac{\delta c}{c}}$$

If you need δa, use $\delta a = a\left(\frac{\delta b}{b} + \frac{\delta c}{c}\right)$.

For dividing, $w = \frac{x}{y}$, the rule is the same as for multiplication:

$$\frac{\delta w}{w} = \frac{\delta x}{x} + \frac{\delta y}{y}$$

For δw, use $\delta w = w\left(\frac{\delta x}{x} + \frac{\delta y}{y}\right)$.

- It is simplest to just remember the single boxed rule for multiplication and division.
- If the expression contains a constant, c, it has δc=0.
- The most important errors in multiplication and division are the largest <u>fractional</u> errors, not absolute errors. This makes sense if you consider that b and c need not have the same units - there is no way to compare the absolute sizes of quantities with different units.

Example:

$$V = IR$$
$$I = 7 \pm 1\ \text{mA}$$
$$R = 20 \pm 2\ \Omega$$

$$V = 140\ \text{mA}\Omega = 140\ \text{mV} = 0.14\ \text{V}$$

$$\frac{\delta V}{V} = \frac{\delta I}{I} + \frac{\delta R}{R} = \frac{1\ \cancel{\text{mA}}}{7\ \cancel{\text{mA}}} + \frac{2\ \cancel{\Omega}}{20\ \cancel{\Omega}} = 0.24$$

$$\delta V = 0.24 \times 0.14\ \text{V} = 0.034\ \text{V} = 34\ \text{mV}$$

Our formula for multiplication indicates that multiplying by a perfectly known constant has no effect on the fractional error of a quantity:

$$F = mg \qquad m = 12 \pm 1\text{kg} \qquad g = 9.8\text{ m/s} \qquad F = 117.6\text{ N}$$

$$\frac{\delta F}{F} = \frac{\delta m}{m} + \frac{\delta g}{g} = \frac{\delta m}{m} \quad \text{since } \delta g = 0$$

$$\text{Then } \frac{\delta F}{F} = \frac{1 \cancel{\text{kg}}}{12 \cancel{\text{kg}}} \quad \text{and} \quad F = 117.6 \pm 9.8\text{ N}$$

The uncertainty $\delta F = F\left(\frac{\delta m}{m}\right) = mg\left(\frac{\delta m}{m}\right) = g\delta m$.

So, δF is just the constant g times δm.

(c) Multiples

If $f = cx + dy + gz$ where c, d, and g are positive or negative constants then,

$\delta(cx) = |c\delta x|$ $\delta(dy) = |d\delta y|$ $\delta(gz) = |g\delta z|$ from the multiplication rule.

From the addition rule,

$$\boxed{\delta f = |c\delta x| + |d\delta y| + |g\delta z|.}$$

(d) Powers

If $f = x^p y^q z^r$ where p, q, and r are positive or negative constants.

$$\frac{\delta f}{f} = \frac{\delta\left(x^p\right)}{x^p} + \frac{\delta\left(y^q\right)}{y^q} + \frac{\delta\left(z^r\right)}{z^r}$$

$$\boxed{\frac{\delta f}{f} = \left|p\frac{\delta x}{x}\right| + \left|q\frac{\delta y}{y}\right| + \left|r\frac{d z}{z}\right|}$$

(e) General

Suppose we want to calculate **f(x),** a function of x, which has uncertainty **δx.** What is the uncertainty in the calculated value f? We simply calculate f at x, and again at $x' = x + \delta x$, then take the absolute value of the difference:

$$\delta f = |\, f(x') - f(x) \,| \quad \text{where } x' = x + \delta x.$$

Example:

$$f(x) = \sin x \qquad x = 30 \pm 1°$$
$$\delta f = |\sin(31°) - \sin(30°)| = |0.515 - 0.500| = 0.015$$

What happens when there is more than one variable? We do the calculation for each variable separately and combine the resulting uncertainties.

$$\delta f(x,y) = |f(x + \delta x, y) - f(x)| + |f(x, y + \delta y) - f(x,y)|$$

(f) When are Errors Negligible?

Errors are only negligible in comparison to something else and in the context of a particular calculation. So it's hard to give general rules, but easier for specific cases. Here's an example of how to think about this question.

You measure a long thin tape (that is, something rectangular). Its length is 10 ± 0.2m, and its width is 2 ± 0.1cm. Which uncertainty is more important? The answer depends on what you want to calculate.

First consider finding the length of the perimeter P of the rectangle formed by the tape.

$$P = 2(L + W)$$

We apply our addition rule:

$$\delta P = 2(\delta L + \delta W)$$

now: $\delta L = 0.2\text{m}$ $\qquad \delta W = 0.1\text{cm} = 0.001\text{m}$

so we can neglect δW since it is much less than δL. We had to put δL and δW into the same units to compare them: 0.2m is much larger that 0.2cm.

Now consider finding the area $A = LW$ of the tape.

The multiplication rule gives:

$$\frac{\delta A}{A} = \frac{\delta L}{L} + \frac{\delta W}{W} = \frac{0.2\cancel{\text{m}}}{10\cancel{\text{m}}} + \frac{0.1\cancel{\text{cm}}}{2.0\cancel{\text{cm}}} = 0.002 + 0.05$$

In this case the uncertainty due to δL is negligible compared to that from δW, the <u>opposite</u> conclusion as for the perimeter calculation! That's because we are multiplying and now need to consider not **δL vs δW**, but instead $\frac{\delta L}{L}$ vs $\frac{\delta W}{W}$.

4. USING UNCERTAINTIES TO COMPARE DATA AND EXPECTATIONS

a) One important question is whether your results agree with what is expected. Let's denote the result by r and the expected value by e. The ideal situation would be $r = e$ or $r - e = 0$. We often use Δ (pronounced "Delta") to denote the difference between two quantities:

$$\boxed{\boldsymbol{\Delta = r - e} \qquad (1)}$$

The standard form for comparison is always *result* - *expected*, so that your difference Δ will be negative if your value is lower than expected, and positive if it is higher than expected.

This comparison must take into account the uncertainty in the observation, and perhaps, in the expected value as well. The data value is $r \pm \delta r$ and the expected value is $e \pm \delta e$. Using the addition/subtraction rule for uncertainties, the uncertainty in $\Delta = r - e$ is just

$$\boxed{\delta\Delta = \delta r + \delta e \quad (2)}$$

Our comparison becomes, "is zero within the uncertainties of the difference Δ?" Which is the same thing as asking if

$$\boxed{|\Delta| \leq \delta\Delta \qquad (3)}$$

Equation (2) and (3) express in algebra the statement "r and e are compatible if their error bars touch or overlap." The combined length of the error bars is given by (2). $|\Delta|$ is magnitude of the separation of r and e . The error bars will overlap (or touch) if r and e are separated by less than (or equal to) the combined length of their error bars, which is what (3) says.

b) Example:

Now we have all we need to do comparisons. For example, if we measure a length of 5.9 ± 0.1cm and expect 6.1 ± 0.1cm, our comparison is:

$$\Delta = r - e = 5.9 \text{ cm} - 6.1 \text{ cm} = -0.2 \text{ cm}$$

while: $$\delta r + \delta e = 0.1 \text{ cm} + 0.1 \text{ cm} = 0.2 \text{ cm}$$

We conclude that our measurement is indeed (barely) consistent with expectations. If we had instead measured 6.4 cm, we would not have been consistent. A good form to display such comparisons is:

R	δr	e	δe	Δ	$\delta\Delta$	compatible?
5.9cm	0.1cm	6.1cm	0.1cm	-0.2cm	0.2cm	YES
6.4cm	0.1cm	6.1cm	0.1cm	+0.3cm	0.2cm	NO
6.2cm	0.2cm	6.1cm	0.1cm	+0.1cm	0.3cm	YES
6.4cm	0.2cm	6.1cm	0	+0.3cm	0.2cm	NO

If only one comparison is to be made, your lab report might contain a sentence like the following: "The measured value was 6.4 ± 0.2cm while the expected value was 6.1 ± 0cm, so the difference is $+0.3 \pm 0.2$cm which means that our measurement was close to, but not compatible with what was expected."