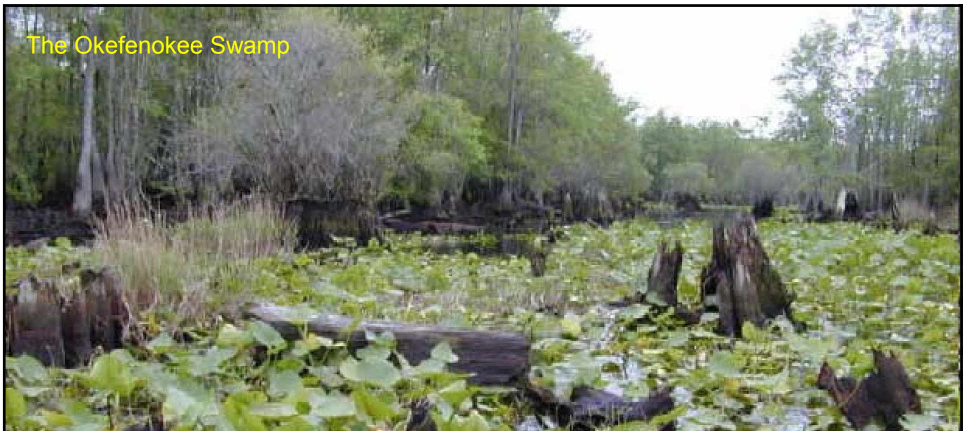
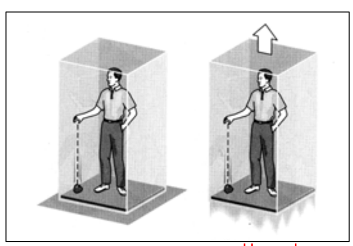
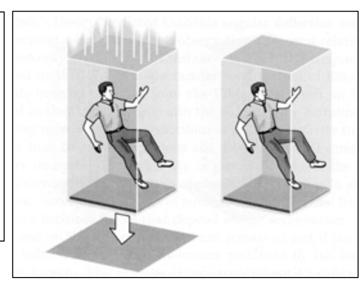


Some books about relativity:

- Taylor & Wheeler *“Spacetime Physics”* = TW1
  (simple presentation, but deep insights)
- Taylor & Wheeler *“Introduction to Black Holes”* = TW2
  (CO-level math, deep insights)
- Weinberg *“Gravitation & Cosmology”*
  (hairy-looking math, but examples of solving field eqn)
- Hartle *“Gravity, An Introduction to Einstein’s General Relativity”*
  (textbook for AST 860)

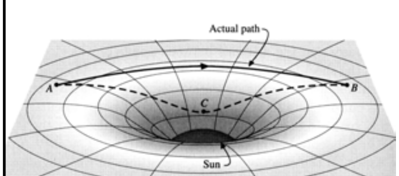
General Relativity (sort of)





Equivalence Principle:

- Can’t tell difference between gravity & acceleration
- …or between freefall & no gravity.
- So *any* experiment should give same answer in either case.



Gravity = Curved space-time

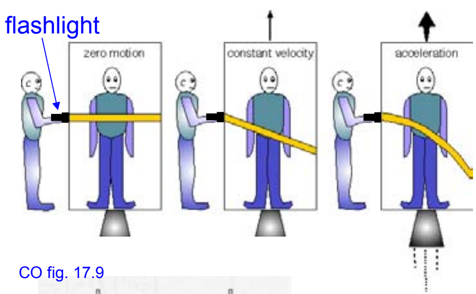
“Weak” equivalence principle:

$$F = ma$$

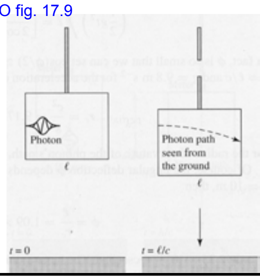
$$F = \frac{GMm}{r^2}$$

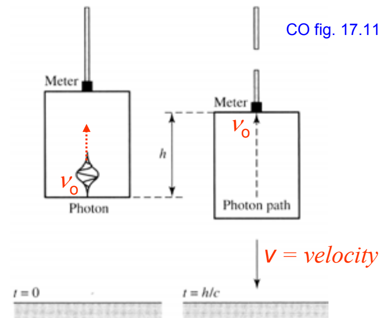
The equivalence principle (plus a vigorous waving of one's hands) shows…

- Curved path of light in gravitational field
- Gravitational Redshift



CO fig. 17.9



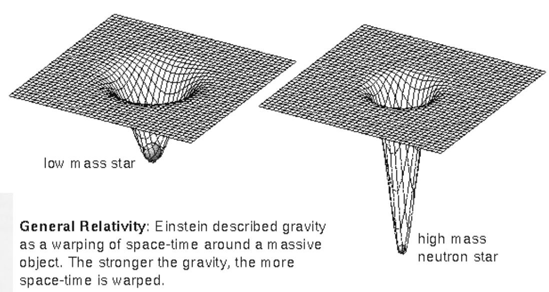


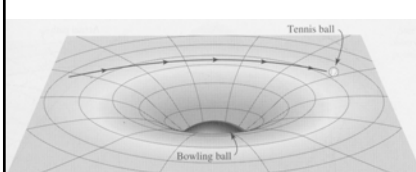
Equiv. Principle ➔ photon frequency unchanged in free-falling lab.

Why doesn't the frequency meter see a *blueshift*?

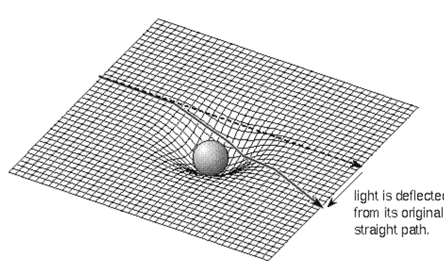
There must be a counteracting *Gravitational Redshift*.

What curves into where?



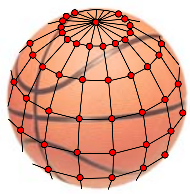


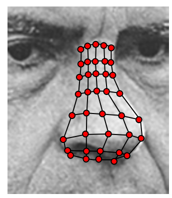
[CO 17.2]



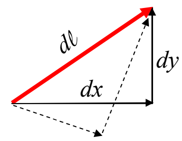
Metrics

$$(d\ell)^2 = (r d\theta)^2 + (r \sin\theta\, d\phi)^2$$





The set of all distances between grids of points, along all different coord. directions, specify shapes.

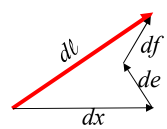


$d\ell^2 = dx^2 + dy^2$
$g_{xx} = g_{yy} = 1$
$g_{xy} = g_{yx} = 0$

$d\ell$ = invariant length

Metric coefficients g_{ij}:

$$d\ell^2 = \sum g_{ij} dx_i dx_j$$



This coord. system needs a more complicated metric, with cross-terms.

Special Relativity

"interval"

$$(ds)^2 = (c\,dt)^2 - (d\ell)^2 = (c\,dt)^2 - [(dx)^2 + (dy)^2 + (dz)^2]$$

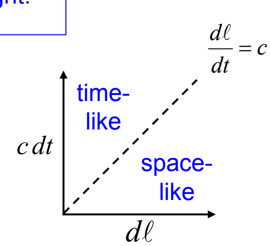
- Proper time:

  If $\Delta\ell = 0$: $\Delta\tau \equiv \frac{\Delta s}{c}$

  Proper distance:

  If $\Delta t = 0$: $\Delta\mathcal{L} = \sqrt{-(\Delta s)^2}$

If $ds = 0$: $c^2 dt^2 = d\ell^2$; $\frac{d\ell}{dt} = c$ → light!



- Metric ←→ Lorentz transform

  [CO pg. 88; TW1 pg. 42]

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}$$

[CO (4.6, 4.9)]

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}$$

Some Metrics

$$ds^2 = \sum g_{ij} dx_i dx_j$$

Flat space-time (special rel. = no gravity):

$$(ds)^2 = (c\,dt)^2 - (d\ell)^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

Flat space-time, spherical coords:

$$(ds)^2 = (c\,dt)^2 - (dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Space filled by uniform matter distribution:

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

Empty space around point source of matter:

$$(ds)^2 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Units, etc. -- to c or not to c:

$$d\tau^2 = \left[1 - \frac{2MG}{r}\right]dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2\,d\theta - r^2\sin^2\theta\,d\varphi^2$$

$ct \rightarrow t$ (Weinberg)

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2$$

$ct \rightarrow t$; $GM/c^2 \rightarrow M$; leave out θ (Taylor & Wheeler)

Vague outline of General Relativity

Stolen from Weinberg, "Gravitation & Cosmology"

Confusion Alarm!!! The $R_{\mu\nu}$ used on this & following slide are *NOT* the Scale Factor!!

Einstein's eqn: $\underbrace{R_{\mu\nu} - g_{\mu\nu}K}_{\text{Curvature}} = \underbrace{-8\pi G T_{\mu\nu}}_{\text{Mass-Energy}}$

$T_{\mu\nu}$ = stress-energy tensor

At each point in space:

$$\begin{pmatrix} E & p_x & p_y & p_z \\ p_x & S_{xx} & S_{xy} & S_{xz} \\ p_y & S_{yx} & S_{yy} & S_{yz} \\ p_z & S_{zx} & S_{zy} & S_{zz} \end{pmatrix}$$

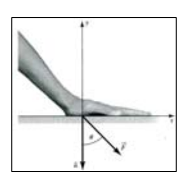
E = energy density
p = momentum flux
S = stress

K = Gaussian curvature
[CO eq. 29.103]

$g_{\mu\nu}$ = components of metric tensor

$R_{\mu\nu}$ = Ricci curvature tensor

"stress" according to Google images



"stress" according to Hartle Fig. 22.3

Einstein's eqn: $R_{\mu\nu} - g_{\mu\nu}K = -8\pi G T_{\mu\nu}$

Stolen from Weinberg, "Gravitation & Cosmology"

Ricci tensor: $R_{\mu\kappa} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\lambda\eta}$ ← Γ= "Affine connection" or "Christoffel symbol"

Example: For spherical symmetry + orthogonal coordinates:

$$d\tau^2 = B(r)\, dt^2 - A(r)\, dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$g_{tt} = -B(r) \quad g_{rr} = A(r) \quad g_{\theta\theta} = r^2 \quad g_{\varphi\varphi} = r^2 \sin^2\theta$$

Einstein's eqn. in empty space: $R_{\mu\nu} = 0$

Non-zero components of Ricci tensor:

Where:
$' = d/dr$
$'' = d^2/dr^2$

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right)$$

$$R_{\theta\theta} = -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)}$$

$$R_{\varphi\varphi} = \sin^2\theta R_{\theta\theta}$$

$$R_{tt} = -\frac{B''(r)}{2A(r)} + \frac{1}{4}\left(\frac{B'(r)}{A(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{B'(r)}{A(r)}\right)$$

$$R_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu$$

Schwarzschild's solution (1916):

Use constraints: At large r, must give Newtonian solution in flat Special Rel. space-time.

$$d\tau^2 = \left[1 - \frac{2MG}{r}\right] dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2\, d\theta - r^2 \sin^2\theta\, d\varphi^2$$

Stolen from Weinberg, "Gravitation & Cosmology"

Schwarzschild's solution (1916):

$$d\tau^2 = \left[1 - \frac{2MG}{r}\right] dt^2 - \underbrace{\left[1 - \frac{2MG}{r}\right]^{-1}} dr^2 - r^2\, d\theta - r^2 \sin^2\theta\, d\varphi^2$$

Blows up at $r = 2MG$

or in another equally good form:

$$\rho \equiv \tfrac{1}{2}\,[r - MG + (r^2 - 2MGr)^{1/2}]$$

$$r = \rho\left(1 + \frac{MG}{2\rho}\right)^2$$

$$d\tau^2 = \frac{(1 - MG/2\rho)^2}{(1 + MG/2\rho)^2}\, dt^2 - \underbrace{\left(1 + \frac{MG}{2\rho}\right)^4} (d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\varphi^2)$$

Stays happy at $r = 2MG$ ➔ $\rho = MG/2$

But what happens with t? Hmmm….

… and there are a bunch more.