Some Metrics

$$ds^2 = \sum g_{ij} dx_i dx_j$$

Flat space-time (special rel. = no gravity):

$$(ds)^2 = (c\,dt)^2 - (d\ell)^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

Flat space-time, spherical coords:

$$(ds)^2 = (c\,dt)^2 - (dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Space filled by uniform matter distribution:

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

Empty space around point source of matter:

$$(ds)^2 = \left(c\,dt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Units, etc. -- to c or not to c:

$$d\tau^2 = \left[1 - \frac{2MG}{r}\right]dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2\,d\theta - r^2\sin^2\theta\,d\varphi^2$$

$ct \rightarrow t$ (Weinberg)

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\phi^2$$

$ct \rightarrow t$; $GM/c^2 \rightarrow M$; leave out θ (Taylor & Wheeler)

Vague outline of General Relativity

Stolen from Weinberg, "Gravitation & Cosmology"

Confusion Alarm!!! The $R_{\mu\nu}$ used on this &

Einstein's eqn: $R_{\mu\nu} - g_{\mu\nu}K = -8\pi G T_{\mu\nu}$

Cu

Same as:

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{pmatrix} - \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix} K = -8\pi G \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix}$$

Same as:

$$R_{11} - g_{11}K = -8\pi G T_{11}$$
$$R_{12} - g_{12}K = -8\pi G T_{12}$$
$$\vdots$$

(etc., for all 16 terms)

$T_{\mu\nu}$ = stress-energy tens

At e

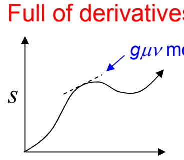
K = Gaussian curvature

[CO eq. 29.103]

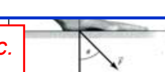
$g_{\mu\nu}$ = components of metric tensor

$R_{\mu\nu}$ = Ricci curvature tensor

Full of derivatives $dg_{\mu\nu}/dx$, etc.



So $dg_{\mu\nu}/dx$ measures curvature.



"stress" according to Hartle Fig. 22.3

Metric for flat space-time, spherical coords:

$$(ds)^2 = (dt)^2 - (dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Stolen from Weinberg, "Gravitation & Cosmology"

Einstein's eqn: $R_{\mu\nu} - g_{\mu\nu}K = -8\pi G T_{\mu\nu}$

In empty space: $R_{\mu\nu} = 0$

For spherical symmetry: $ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)$

Unknown functions, allow space to be curved

Non-zero components of Einstein Eqn:

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right) = 0$$

$$R_{\theta\theta} = -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)} = 0$$

$$R_{\varphi\varphi} = \sin^2\theta\,R_{\theta\theta} = 0$$

$$R_{tt} = -\frac{B''(r)}{2A(r)} + \frac{1}{4}\left(\frac{B'(r)}{A(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{B'(r)}{A(r)}\right) = 0$$

Where:
$' = d/dr$
$'' = d^2/dr^2$

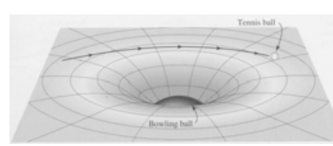
Schwarzschild's solution (1916):

$$ds^2 = \left[1 - \frac{2MG}{r}\right]dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2\,d\theta - r^2\sin^2\theta\,d\varphi^2$$

Black Holes & the Schwarzschild Metric

Simplified metric, from Taylor & Wheeler2 (no c^2, no G, no θ):

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2$$



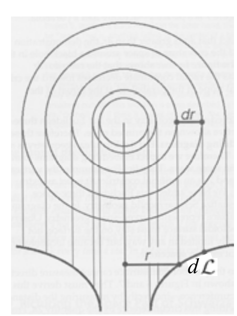
- Schwarzschild radius: $r = 2M$
- *Schwarzschild coordinates*
  - reconstructed as if seen from a point where space is flat.
- Metric for observer sitting on a shell at r

Proper time:

$$d\tau = dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt$$

Proper distance:

$$d\mathcal{L} = dr_{\text{shell}} = \frac{dr}{\left(1 - \frac{2M}{r}\right)^{1/2}}$$

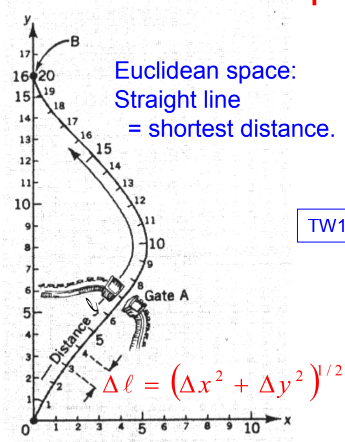


➔ Observer measures things in locally flat space-time:

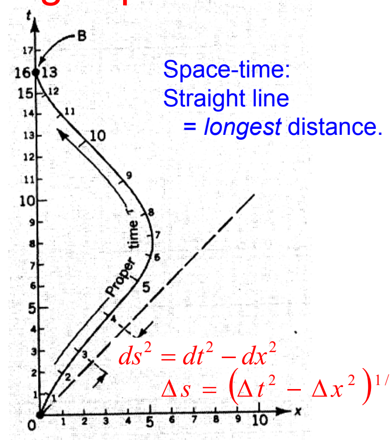
$$ds^2 = dt_{\text{shell}}^2 - dr_{\text{shell}}^2 - r^2 d\phi^2$$

- Free-falling astronaut:
  - Metric = flat space-time in local region.
  - Time on wristwatch = $\tau = s$.

World-Lines = path through space-time



TW1, Fig. 1-19



Geodesics

- Light always follows *null geodesic*, with $\Delta s = \Delta \tau = 0$
- In free-fall, world-line of objects with mass = extremum of $\int ds$ (= $\int d\tau$)
  - Normally a maximum
    ➔ *observer travelling on Geodesic sees max. time pass*

$$\frac{ds}{d?} = 0$$

Geodesics

Dubious example in CO (pgs. 632-633)

$$(ds)^2 = \left(c\,dt\sqrt{1 - 2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1 - 2GM/rc^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

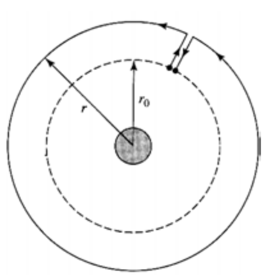
A satellite in circular orbit.

Assume:

- Circular orbit ➔ $dr = 0$
- In plane where $d\phi = 0$
- Moving at "specified angular speed" ω ➔ $d\theta = \omega\,dt$

$$(ds)^2 = \left[\left(c\sqrt{1 - 2GM/rc^2}\right)^2 - r^2\omega^2\right]dt^2 = \left(c^2 - \frac{2GM}{r} - r^2\omega^2\right)dt^2$$

$$\Delta s = \int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\,dt.$$



Find r that makes Δs be an extremum:

$$\frac{d}{dr}(\Delta s) = \frac{d}{dr}\left(\int_0^{2\pi/\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\,dt\right) = 0$$

$$\frac{d}{dr}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2} = 0$$

$$\frac{2GM}{r^2} - 2r\omega^2 = 0$$

$$v = r\omega = \sqrt{\frac{GM}{r}}$$

A familiar Newtonian result!

Geodesics

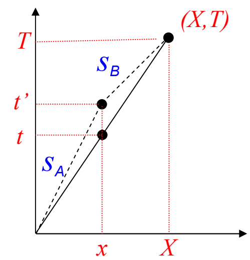
Better Example: Conservation of Energy in Special Relativity

(see TW2, pp. 109-111). Metric is: $\Delta s^2 = \Delta t^2 - \Delta x^2$

Simplify notation: Let $s_A = \Delta s$ between points (0,0) and (t, x), etc.

$$s_A^2 = t^2 - x^2$$
$$s_A = \left(t^2 - x^2\right)^{1/2}$$
$$\frac{ds_A}{dt} = \frac{t}{\left(t^2 - x^2\right)^{1/2}} = \frac{t_A}{s_A}$$

$$s_B^2 = (T-t)^2 - (X-x)^2$$
$$s_B = \left((T-t)^2 - (X-x)^2\right)^{1/2}$$
$$\frac{ds_B}{dt} = -\frac{(T-t)}{s_B} = -\frac{t_B}{s_B}$$



$$\frac{ds}{dt} = \frac{ds_A}{dt} + \frac{ds_B}{dt} = \frac{t_A}{s_A} - \frac{t_B}{s_B} = 0$$

Here is where we find the extremum!

$$\frac{t_A}{s_A} = \frac{t_B}{s_B} = any\,\frac{\Delta t}{\Delta s}$$

= constant of motion. *Let's call it "energy/mass".*

$E = m\, \Delta t/\Delta s = m\, dt/ds$

Similar argument to get momentum: $p = m\, dx/ds$

Homework for next class – find definition of energy for Schwarzschild metric.

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2$$