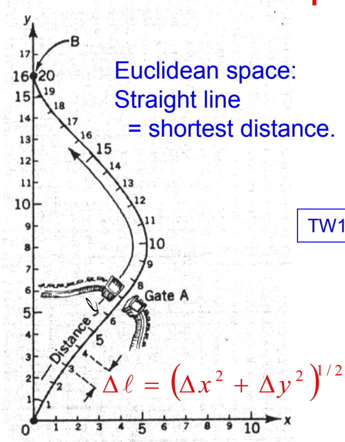
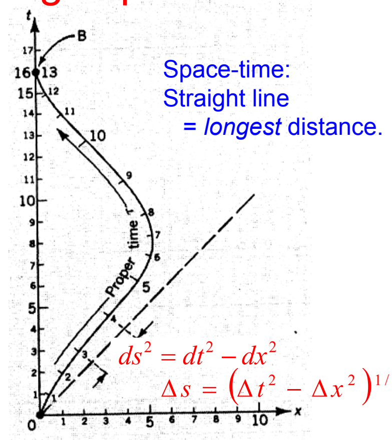
World-Lines = path through space-time



TW1, Fig. 1-19



Geodesics

- Light always follows *null geodesic*, with $\Delta s = \Delta\tau = 0$
- In free-fall, world-line of objects with mass = extremum of $\int ds$ $\left(= \int d\tau\right)$
  - Normally a maximum
    ➔ *observer travelling on Geodesic sees max. time pass*

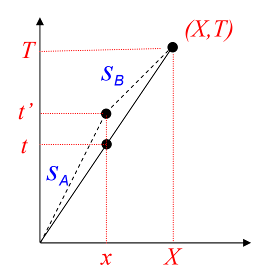
$$\frac{ds}{d\,?} = 0$$

Geodesics

Better Example: Conservation of Energy in Special Relativity

(see TW2, pp. 109-111). Metric is: $\Delta s^2 = \Delta t^2 - \Delta x^2$

Simplify notation: Let $s_A = \Delta s$ between points (0,0) and (t, x), etc.



$$s_A^2 = t^2 - x^2$$
$$s_A = \left(t^2 - x^2\right)^{1/2}$$
$$\frac{ds_A}{dt} = \frac{t}{\left(t^2 - x^2\right)^{1/2}} = \frac{t_A}{s_A}$$

$$s_B^2 = (T-t)^2 - (X-x)^2$$
$$s_B = \left((T-t)^2 - (X-x)^2\right)^{1/2}$$
$$\frac{ds_B}{dt} = -\frac{(T-t)}{s_B} = -\frac{t_B}{s_B}$$

$$\frac{ds}{dt} = \frac{ds_A}{dt} + \frac{ds_B}{dt} = \frac{t_A}{s_A} - \frac{t_B}{s_B} = 0$$

Here is where we find the extremum!

$$\frac{t_A}{s_A} = \frac{t_B}{s_B} = any\,\frac{\Delta t}{\Delta s}$$

= constant of motion. *Let's call it "energy/mass".*

$$E = m\ \Delta t/\Delta s = m\ dt/ds$$

Similar argument to get momentum: $p = m\ dx/ds$

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2$$

Homework for next class – find definition of energy for Schwarzschild metric.

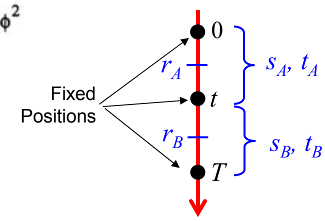
Find definition of energy for Schwarzschild metric:

See TW2, pg. 3-7

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\phi^2$$

$$\Delta s^2 = \left(1-\frac{2M}{r}\right)\Delta t^2 + const$$

$$s^2 = \left(1-\frac{2M}{r}\right)t^2 + const$$



$$s_A = \left[\left(1-\frac{2M}{r_A}\right)t^2 + const\right]^{1/2}$$

$$\frac{ds_A}{dt} = \frac{1}{2}\left[\left(1-\frac{2M}{r_A}\right)t^2 + const\right]^{-1/2}\left(1-\frac{2M}{r_A}\right)2t$$

$$\frac{ds_A}{dt} = \left(1-\frac{2M}{r_A}\right)\frac{t_A}{s_A}$$

$$s_B = \left[\left(1-\frac{2M}{r_B}\right)(T-t)^2 + const\right]^{1/2}$$

$$\frac{ds_B}{dt} = -\frac{1}{2}\left[\left(1-\frac{2M}{r_B}\right)(T-t)^2 + const\right]^{-1/2}\left(1-\frac{2M}{r_B}\right)2(T-t)$$

$$\frac{ds_B}{dt} = -\left(1-\frac{2M}{r_B}\right)\frac{(T-t)}{s_B} = -\left(1-\frac{2M}{r_B}\right)\frac{t_B}{s_B}$$

$$\left(1-\frac{2M}{r_A}\right)\frac{t_A}{s_A} = \left(1-\frac{2M}{r_B}\right)\frac{t_B}{s_B} = \left(1-\frac{2M}{r}\right)\frac{\Delta t}{\Delta s} = \boxed{\left(1-\frac{2M}{r}\right)\frac{dt}{ds} = \frac{E}{m}}$$

The Answer

A handy little shortcut for doing this sort of derivative

$$s_A^2 = \left(1-\frac{2M}{r_A}\right)t^2 + const.$$

$$2s_A ds_A = \left(1-\frac{2M}{r_A}\right)2t\,dt$$

$$\frac{ds_A}{dt} = \left(1-\frac{2M}{r_A}\right)\frac{t}{s_A}$$

Falling into a Black Hole

- Schw. conservation of energy: $\frac{E}{m} = \left(1-\frac{2M}{r}\right)\frac{dt}{ds} = 1$ ($E = m$ at $r = \infty$)

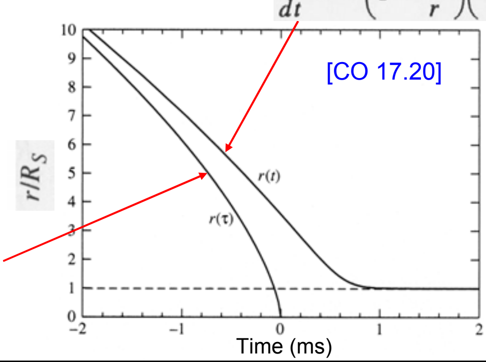
$$\left(\frac{dt}{ds}\right) = \frac{1}{\left(1-\frac{2M}{r}\right)}$$

- Schw. metric

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2\overset{0}{\cancel{d\phi^2}}$$

Eq. of motion for freefall in Schw. Coords:

$$\frac{dr}{dt} = -\left(1-\frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2}$$



In time units of free-falling astronaut:

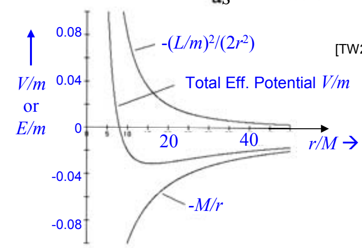
$$\frac{dr}{ds} = \frac{dr}{dt}\frac{dt}{ds} = -\left(\frac{2M}{r}\right)^{1/2}$$

Orbits around black holes
(from TW2)

- Schw. metric $ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\phi^2$
- Schw. conservation of energy: $\frac{E}{m} = \left(1-\frac{2M}{r}\right)\frac{dt}{ds}$
- Schw. conservation of angular mom: $r^2\frac{d\phi}{ds} = \text{constant}$ $\quad \frac{L}{m} = r^2\frac{d\phi}{ds}$
- Effective potential
  - Newtonian

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \left[-\frac{M}{r} + \frac{(L/m)^2}{2r^2}\right]$$

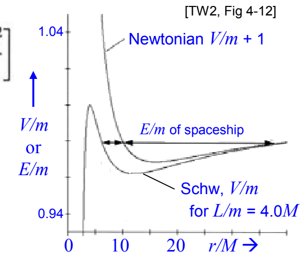
$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{E}{m} - \frac{V(r)}{m}$$

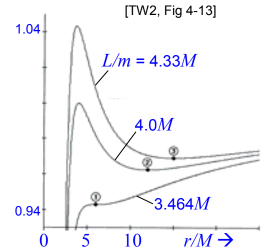


  - Schwarzschild

$$\left(\frac{dr}{ds}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1-\frac{2M}{r}\right)\left[1+\frac{(L/m)^2}{r^2}\right]$$

$$\left(\frac{dr}{ds}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(\frac{V}{m}\right)^2$$





- Precession of perihelion
- Innermost stable orbit

Spinning Black Holes

Notation:
No G, no c

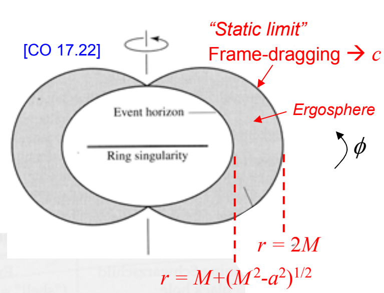
Kerr metric (1963)

$d\phi d\tau$ cross term → "frame dragging"

$$ds^2 = -\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\phi^2$$

where $a \equiv J/M, \quad \rho^2 \equiv r^2 + a^2\cos^2\theta, \quad \Delta \equiv r^2 - 2Mr + a^2$

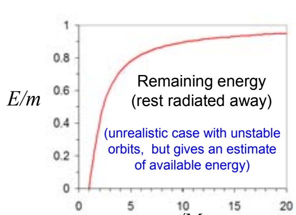
J = Angular Momentum

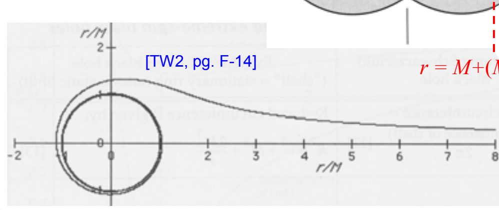


Maximal spin: $J_{max} = M^2$ (or $= GM^2/c^2$ in CO units)

- Usually ~ the case.
- Then Event Horizon in equatorial plane is at $r=M$

Infalling particle with no angular momentum:





Both plots for equatorial plane only

Metric for flat space-time, spherical coords:

Stolen from Weinberg, "Gravitation & Cosmology"

$$(ds)^2 = (dt)^2 - (dr)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

Einstein's eqn: $R_{\mu\nu} - g_{\mu\nu}K = -8\pi G T_{\mu\nu}$

In empty space: $R_{\mu\nu} = 0$

For spherical symmetry: $ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)$

Unknown functions, allow space to be curved

Non-zero components of Einstein Eqn:

Where:
$' = d/dr$
$'' = d^2/dr^2$

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right) = 0$$

$$R_{\theta\theta} = -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)} = 0$$

$$R_{\varphi\varphi} = \sin^2\theta R_{\theta\theta} = 0$$

$$R_{tt} = -\frac{B''(r)}{2A(r)} + \frac{1}{4}\left(\frac{B'(r)}{A(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{B'(r)}{A(r)}\right) = 0$$

Schwarzschild's solution (1916):

$$ds^2 = \left[1 - \frac{2MG}{r}\right]dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2\,d\theta - r^2\sin^2\theta\,d\varphi^2$$

GR and Cosmology

[CO] notation, with c^2

- Robertson-Walker metric:

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

- Cosmological Principle *requires* the RW metric.
  - RW is the only possible metric for a spherically symmetric homogeneously expanding or contracting space-time (Weinberg, pg. 403)
    - *e.g.* it is also the appropriate metric for the interior of a homogeneously collapsing star.
- So then the free parameters in the metric are k and $R(t)$.
- Energy momentum tensor must have the "perfect fluid" form:

Back to Weinberg's notation, without c^2

$$T_{\mu\nu} = p g_{\mu\nu} + (p+\rho)U_\mu U_\nu$$

where ρ and p are functions of t alone, and $U^t = 1 \qquad U^i = 0$

- The non-zero components of the Einstein equation then reduce to

$\dot{R} = \dfrac{dR}{dt}$ etc.

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2$$ Friedmann eqn.[29.10]

$$\dot{p}R^3 = \frac{d}{dt}\{R^3[\rho + p]\}$$

$$\frac{d}{dR}(\rho R^3) = -3pR^2$$ Form of fluid eqn.[29.50]

Stolen from Weinberg, "Gravitation & Cosmology"

Curved Spaces & the Robertson-Walker Metric

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

- R-W metric: most general solution for universe obeying Cosmological Principle.
  - Homogeneous & Isotropic
  - Smooth distribution of matter.
  - Same everywhere at any given time.
- Curvature

Defined incorrectly in 10/21 notes.

$$K \equiv \frac{k}{R^2(t)}$$

$$k \equiv \frac{1}{\Re_0^2}$$

$\Re_0 =$ Present radius of curvature (meters)

  - Can be found from local measurements
    - By bug on sphere