# Curved Spaces & the Robertson-Walker Metric

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

- R-W metric: most general solution for universe obeying Cosmological Principle.
  - Homogeneous & Isotropic
  - Smooth distribution of matter.
  - Same everywhere at any given time.
- Curvature

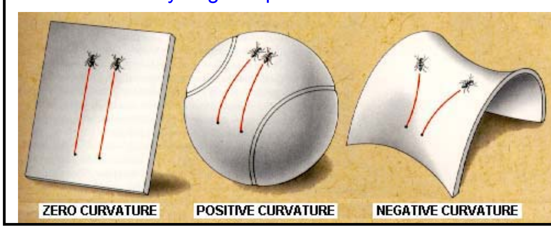
*Defined incorrectly in 10/21 notes.*

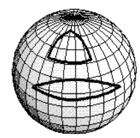
$$K \equiv \frac{k}{R^2(t)}$$

$$k \equiv \frac{1}{\Re_0^2}$$

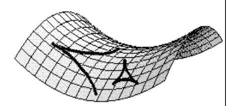
$\Re_0$ = Present radius of curvature (meters)

  - Can be found from local measurements
    - By bug on sphere

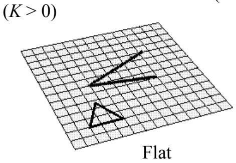


Positive Curvature ($K > 0$)



Negative Curvature ($K < 0$)



Flat ($K = 0$)

# Geometry of a 2D Spherical Surface

$$r = R\sin\theta$$

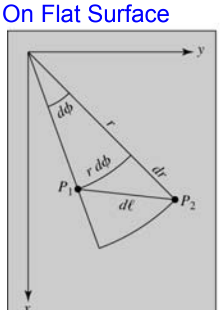
$$dr = R\cos\theta\,d\theta$$

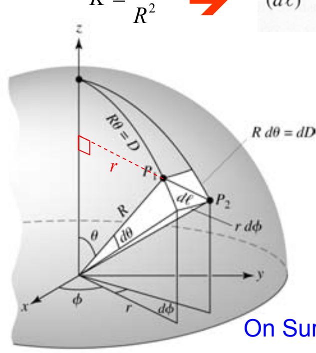
$$R\,d\theta = \frac{dr}{\cos\theta} = \frac{R\,dr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - r^2/R^2}} \;\Rightarrow\; (d\ell)^2 = \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\,d\phi)^2$$

$$(d\ell)^2 = (dD)^2 + (r\,d\phi)^2 = (R\,d\theta)^2 + (r\,d\phi)^2$$

$$K = \frac{1}{R^2} \;\Rightarrow\; (d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\phi)^2$$

On Flat Surface




On Surface of Ball

# Geometry of a 2D Spherical Surface

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$$dr = R\cos\theta\, d\theta$$

$$R\,d\theta = \frac{dr}{\cos\theta} = \frac{R\,dr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - r^2/R^2}} \;\rightarrow\; (d\ell)^2 = (dD)^2 + (r\,d\phi)^2 = (R\,d\theta)^2 + (r\,d\phi)^2$$

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\,d\phi)^2$$

$$K = \frac{1}{R^2} \;\rightarrow\; (d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\phi)^2$$

To get R-W metric:

- Add another dimension

$$(d\ell)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2$$

- Add time

$$(ds)^2 = (c\,dt)^2 - \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 - (r\,d\theta)^2 - (r\sin\theta\,d\phi)^2$$

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1 - k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

where $r(t) = R(t)\varpi$ and $K(t) \equiv \dfrac{k}{R^2(t)}$

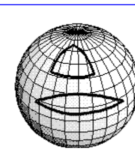
# Geometry of a 3D Surface (our Universe) in an Expanding 4D Space

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1 - k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

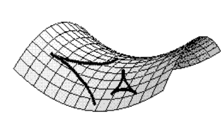
Area of sphere at co-moving distance $\varpi$: $A = 4\pi\varpi^2$

$r(t) = R(t)\varpi$

$K(t) \equiv \dfrac{k}{R^2(t)}$

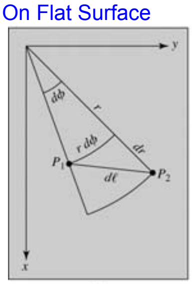


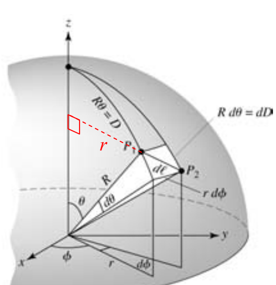
Positive Curvature ($K > 0$)



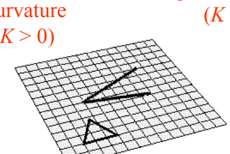
Negative Curvature ($K < 0$)

On Flat Surface




On Surface of Ball



Flat ($K = 0$)

# Dynamics of a 3D Surface (our Universe) in an Expanding 4D Space

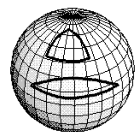
$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

$$r(t) = R(t)\,\varpi$$

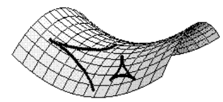
$$K(t) \equiv \frac{k}{R^2(t)}$$

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2$$

Friedmann Equation

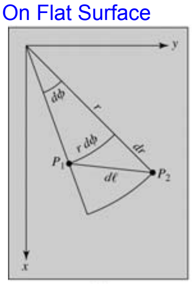


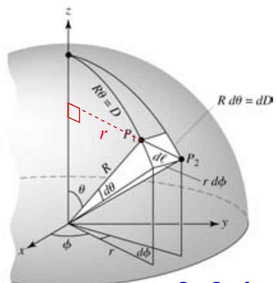
Positive Curvature ($K > 0$)



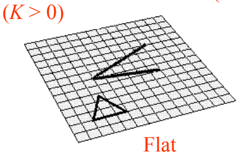
Negative Curvature ($K < 0$)

On Flat Surface





On Surface of Ball



Flat ($K = 0$)

# Dynamics of a 3D Surface (our Universe) in an Expanding 4D Space

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

$$r(t) = R(t)\,\varpi$$

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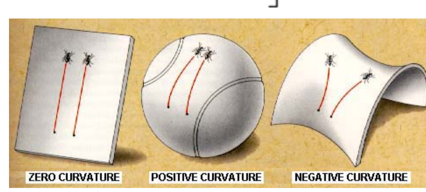
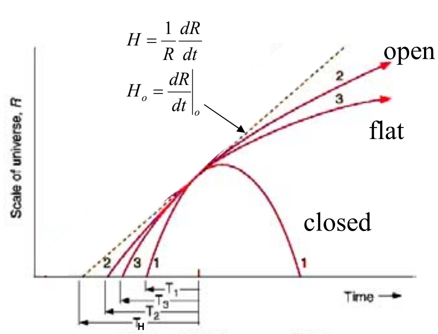
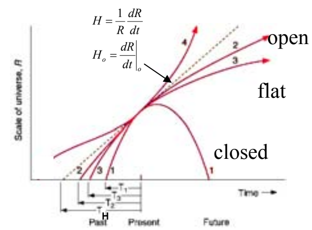



- Dynamics and curvature both due to mass-energy density.
- For Friedmann Eqn *without* Cosmological Constant:

| Density | Curvature | Dynamics |
|---|---|---|
| $\rho_o < \rho_{c,o}$ | Negative | Expands forever |
| $\rho_o = \rho_{c,o}$ | Flat | Oozes to stop at $t = \infty$ |
| $\rho_o > \rho_{c,o}$ | Positive | Collapses back |

# Sneak Preview




Accelerating Universe

- Hubble's law:

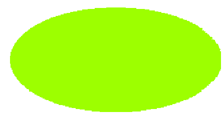
  $v(t) = H(t)\, r(t)$

- Lookback time ➔ for more distant objects, we measure $H(t)$ at earlier $t$.
- If gravity constantly slows expansion, expect larger $H$ at earlier $t$.
- Late 1990s: Type Ia SNe showed $H(t)$ is currently increasing with time.

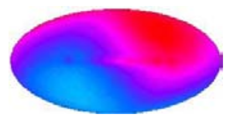
Flat Universe

- CMB fluctuations
- 1 part in $10^5$ amplitude
- COBE: angular resolution too low
- WMAP has higher angular resolution.
- Fluctuations have known physical size.
- Angular size depends on geometry of universe.

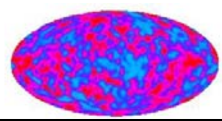
COBE results



Blue = 0°K
Red = 4°K



Blue = 2.724°K
Red = 2.732°K
Dipole Anistropy
➔motion of Sun through Universe.



After removing dipole
Red – blue = 0.0002°K

# The Cosmological Constant

Einstein's Eqn:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

One last possible term. $\Lambda$ = arbitrary constant.

The "complete" Friedmann Eqn:

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2$$

## $\Lambda$ acts as outward force:

Newtonian version: $\frac{1}{2}mv^2 - G\frac{M_r m}{r} - \frac{1}{6}\Lambda m c^2 r^2 = -\frac{1}{2}mkc^2\varpi^2$

Define a potential:

$$U_\Lambda \equiv -\frac{1}{6}\Lambda m c^2 r^2 \quad ➔ \quad \mathbf{F}_\Lambda = -\frac{\partial U_\Lambda}{\partial r}\hat{\mathbf{r}} = \frac{1}{3}\Lambda m c^2 r\,\hat{\mathbf{r}} \qquad [29.109]$$

The acceleration Eqn:

$$\frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{\rm rel} + \frac{3(P_m + P_{\rm rel})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right\} R \qquad [29.112]$$

*Psst*… is it a constant?

# Equation of state [CO pp. 1161-1162]

- Relation between $P$, $R$ and $\rho$

Define: $\rho = \rho_o R^{-3(1+w)}$

Fluid eqn (29.50): $\dfrac{d(R^3\rho)}{dt} = -\dfrac{P}{c^2}\dfrac{d(R^3)}{dt}$

[29.52] $P = wu = w\rho c^2$

$\rightarrow$

Matter: $w = 0$ $\quad \rho \propto R^{-3}$

Radiation: $w = 1/3$ $\quad \rho \propto R^{-4}$

Cosm, Const: $w = -1$ $\quad \rho \propto R^{0}$

# Cosmological Constant as a "Negative Pressure"

Friedmann Eq. with Cosmological Constant:

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2$$

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0} \rightarrow \left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{\rm rel} + \rho_\Lambda)\right] R^2 = -kc^2 \quad [29.114]$$

$$[29.111] \quad \frac{d(R^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt} \rightarrow \frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{\rm rel} + \frac{3(P_m + P_{\rm rel})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right\} R$$

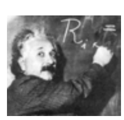
Negative pressure (29.115)

$$P_\Lambda = -\rho_\Lambda c^2 \rightarrow \frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{\rm rel} + \rho_\Lambda + \frac{3(P_m + P_{\rm rel} + P_\Lambda)}{c^2}\right]\right\} R$$

*Psst*... is it a constant? $P = wu = w\rho c^2$ Is $w$ really -1? [29.116]

# A slight renaming….

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2$$



COSMOLOGICAL CONSTANT

# A slight renaming….

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2$$ [29.108]

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0}$$ [29.113]

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\rho_m + \rho_{rel} + \rho_\Lambda\right)\right)R^2 = -kc^2$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda c^2}{3H^2}$$

[29.9]

$$\Omega \equiv \Omega_m + \Omega_{\rm rel} + \Omega_\Lambda$$ [29.119]

$$\left(H^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2 \rightarrow H^2(1-\Omega)R^2 = -kc^2$$ Friedmann Eqn

$$H_0^2(1-\Omega_0) = -kc^2$$ [29.121]

$$H_0^2(1-\Omega_0) = -kc^2$$ [29.121]

The basic WMAP result: $k$ = 0

$$\rightarrow [\Omega_0]_{\rm WMAP} = 1.02 \pm 0.02$$

$$\Omega_0 = \Omega_{m,0} + \Omega_{\rm rel,0} + \Omega_{\Lambda,0} = 1$$

$$[\Omega_{m,0}]_{\rm WMAP} = 0.27 \pm 0.04$$

$$\Omega_{\rm rel,0} = 8.24 \times 10^{-5}$$

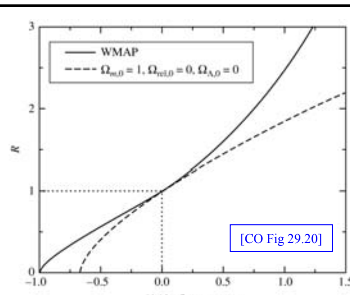
$$[\Omega_{\Lambda,0}]_{\rm WMAP} = 0.73 \pm 0.04$$

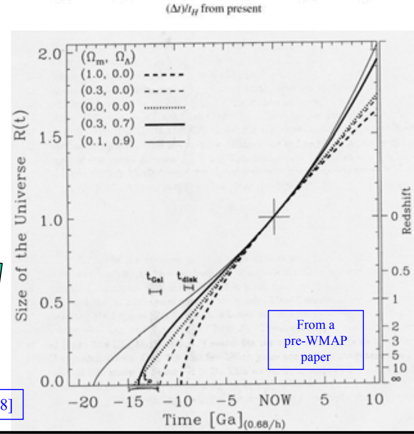
Still another form of Friedmann Eqn:

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\rho_m + \rho_{rel} + \rho_\Lambda\right)\right)R^2 = -kc^2$$

Solution for $k$ = 0:

$$t = \sqrt{\frac{3}{8\pi G}}\int_0^R \frac{R'\,dR'}{\sqrt{\rho_{m,0}R' + \rho_{\rm rel,0} + \rho_{\Lambda,0}R'^4}}.$$ [29.128]



$$\left(H^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2$$

$$H^2(1-\Omega)R^2 = -kc^2$$

$$H_0^2(1-\Omega_0) = -kc^2$$

# Some Universes

Open *vs.* Closed:

$$k = 0 \;\Rightarrow\; \Omega_0 = \Omega_{m,0} + \underbrace{\cancel{\Omega_{\mathrm{rel},0}}}_{\sim 0} + \Omega_{\Lambda,0} = 1$$

Accelerating *vs.* Decelerating:

For $\Lambda = 0$:

$$q(t) = -\frac{R(t)\,[d^2R(t)/dt^2]}{[dR(t)/dt]^2} = \frac{1}{2}\Omega(t)$$

$$q_0 = \frac{1}{2}\Omega_0$$

Deceleration parameter

For $\Lambda = 0$:
$q_o = 0$ empty
$< 0.5$ open
$= 0.5$ flat
$> 0.5$ closed

For $\Lambda \neq 0$: $$q(t) = \frac{1}{2}\sum_i (1+3w_i)\Omega_i(t)$$

$$q(t) = \frac{1}{2}\Omega_m(t) + \underbrace{\cancel{\Omega_{\mathrm{rel}}(t)}}_{\sim 0} - \Omega_\Lambda(t)$$

Expands Forever *vs.* Recollapses:

Does $dR/dt$ ever = 0?

See [29.135]