Equation of state [CO pp. 1161-1162]

- Relation between P, R and ρ

Define: $\rho = \rho_o R^{-3(1+w)}$

Fluid eqn (29.50): $\frac{d(R^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt}$

[29.52] $P = wu = w\rho c^2$

→

Matter: $w = 0$ $\rho \propto R^{-3}$

Radiation: $w = 1/3$ $\rho \propto R^{-4}$

Cosm, Const: $w = -1$ $\rho \propto R^{0}$

Cosmological Constant as a "Negative Pressure"

Friedmann Eq. with Cosmological Constant:

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2$$

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0} \rightarrow \left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G(\rho_m + \rho_{\rm rel} + \rho_\Lambda)\right]R^2 = -kc^2$$ [29.114]

[29.111]

$$\frac{d(R^3\rho)}{dt} = -\frac{P}{c^2}\frac{d(R^3)}{dt} \rightarrow \frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\left[\rho_m + \rho_{\rm rel} + \frac{3(P_m + P_{\rm rel})}{c^2}\right] + \frac{1}{3}\Lambda c^2\right\}R$$

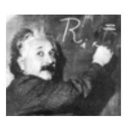
Negative pressure (29.115)

$$P_\Lambda = -\rho_\Lambda c^2 \rightarrow \frac{d^2R}{dt^2} = \left\{-\frac{4}{3}\pi G\underbrace{\left[\rho_m + \rho_{\rm rel} + \rho_\Lambda + \frac{3(\cancelto{0}{P_m} + \cancelto{\rho_{rel}c^2/3}{P_{\rm rel}} + \cancelto{-\rho_\Lambda c^2}{P_\Lambda})}{c^2}\right]}_{[\rho_m\ +\ 2\rho_{rel}\ -\ 2\rho_\Lambda]}\right\}R$$

Psst… is it a constant? $P = wu = w\rho c^2$ Is w really -1?

A slight renaming….

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2$$



COSMOLOGICAL CONSTANT

A slight renaming….

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2$$ [29.108]

$$\rho_\Lambda \equiv \frac{\Lambda c^2}{8\pi G} = \text{constant} = \rho_{\Lambda,0}$$ [29.113]

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\rho_m + \rho_{rel} + \rho_\Lambda\right)\right) R^2 = -kc^2$$

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda c^2}{3H^2}$$

[29.9]

$$\Omega \equiv \Omega_m + \Omega_{\text{rel}} + \Omega_\Lambda$$ [29.119]

$$\left(H^2 - \frac{8}{3}\pi G\rho\right) R^2 = -kc^2 \rightarrow H^2(1-\Omega)R^2 = -kc^2$$ Friedmann Eqn

$$H_0^2(1-\Omega_0) = -kc^2$$ [29.121]

$$H_0^2(1-\Omega_0) = -kc^2$$ [29.121]

The basic WMAP result: $k = 0$

$$\rightarrow [\Omega_0]_{\text{WMAP}} = 1.02 \pm 0.02$$

$$\Omega_0 = \Omega_{m,0} + \Omega_{\text{rel},0} + \Omega_{\Lambda,0} = 1$$

$$[\Omega_{m,0}]_{\text{WMAP}} = 0.27 \pm 0.04$$

$$\Omega_{\text{rel},0} = 8.24 \times 10^{-5}$$

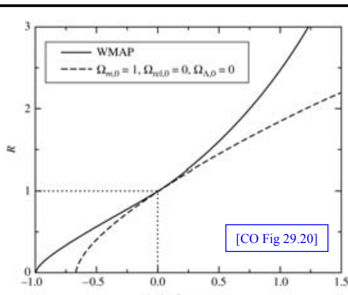
$$[\Omega_{\Lambda,0}]_{\text{WMAP}} = 0.73 \pm 0.04$$

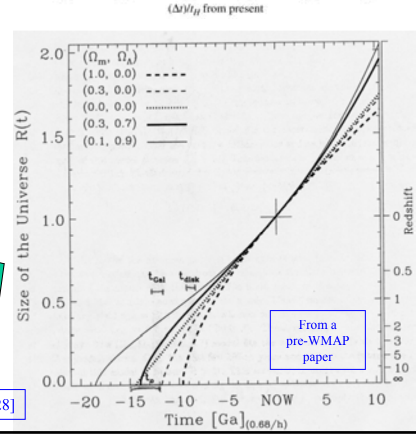
Still another form of Friedmann Eqn:

$$\left(\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\rho_m + \rho_{rel} + \rho_\Lambda\right)\right) R^2 = -kc^2$$

Solution for $k = 0$:

$$t = \sqrt{\frac{3}{8\pi G}} \int_0^R \frac{R'\,dR'}{\sqrt{\rho_{m,0}R' + \rho_{\text{rel},0} + \rho_{\Lambda,0}R'^4}}.$$ [29.128]





$$\left(H^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2$$

$$H^2(1-\Omega)R^2 = -kc^2$$

$$H_0^2(1-\Omega_0) = -kc^2$$

Some Universes

Open *vs.* Closed:

$$k = 0 \Rightarrow \Omega_0 = \Omega_{m,0} + \cancelto{\sim 0}{\Omega_{\mathrm{rel},0}} + \Omega_{\Lambda,0} = 1$$

Accelerating *vs.* Decelerating:

$$q(t) = -\frac{R(t)\,[d^2R(t)/dt^2]}{[dR(t)/dt]^2}$$

Deceleration parameter

For $\Lambda = 0$: $= \frac{1}{2}\Omega(t)$

$$q_0 = \frac{1}{2}\Omega_0$$

For $\Lambda = 0$:
$q_o = 0$ empty
< 0.5 open
$= 0.5$ flat
> 0.5 closed

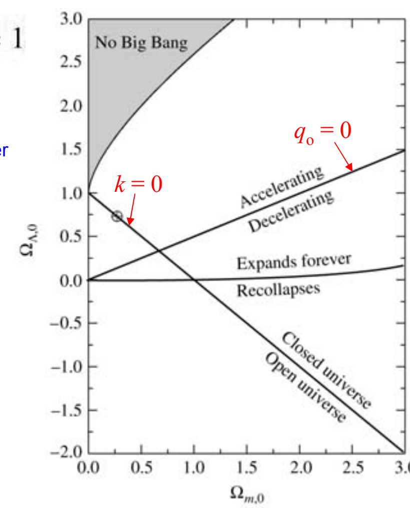
For $\Lambda \neq 0$: $q(t) = \frac{1}{2}\sum_i (1+3w_i)\Omega_i(t)$

$$q(t) = \frac{1}{2}\Omega_m(t) + \cancelto{\sim 0}{\Omega_{\mathrm{rel}}(t)} - \Omega_\Lambda(t)$$

Expands Forever *vs.* Recollapses:

Does dR/dt ever = 0?

See [29.135]



[29.4] Observational Cosmology

$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\,d\theta)^2 + (\varpi\sin\theta\,d\phi)^2\right]$$

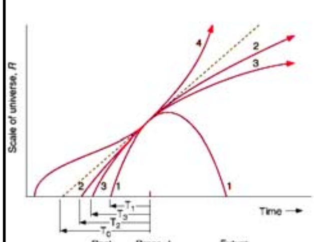
$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right]R^2 = -kc^2$$

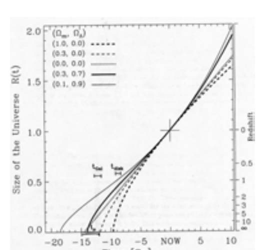
$$H_0^2(1-\Omega_0) = -kc^2$$

- Some theoretical parameter sets:
  - *R(t)* vs. *t*
  - $\Omega_{\Lambda,o}$ vs. $\Omega_{m,o}$
  - Curvature *k, dR/dt, d²R/dt²*

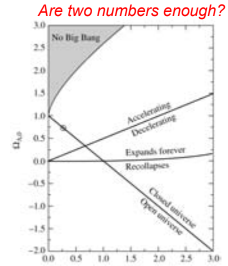
- But what can we actually *measure* that will tell us which universe we live in?

  As a function of *z*:
  - Apparent mag. of standard candles.
  - Angular sizes.
  - Space density of galaxies.

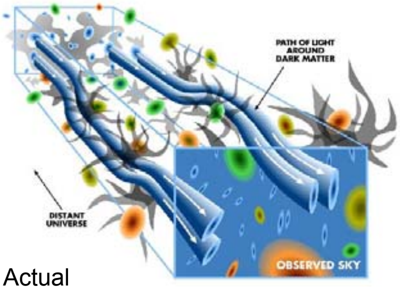




Are two numbers enough?



The paths of photons through space-time



Actual
(sketched in 3D space)

Idealized
(R-W metric)

$$(\cancelto{0}{ds})^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\cancelto{0}{\varpi\, d\theta})^2 + (\cancelto{0}{\varpi \sin\theta\, d\phi})^2\right]$$

$$\int_{t1}^{t2} \frac{c\,dt}{R(t)} = -\int_{\varpi 1}^{\varpi 2} \frac{d\varpi}{\sqrt{1-k\varpi^2}} = \int_{\varpi 2}^{\varpi 1} \frac{d\varpi}{\sqrt{1-k\varpi^2}}$$

Take neg. square root so that
ϖ will decrease with increasing t

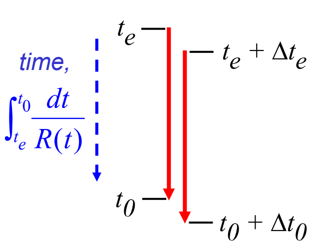
Redshift and Cosmological Time Dilation

(See pg. 1200)

$$(ds)^2 = 0 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + \cdots\right.$$

For two radially travelling wave crests:

$$\int_{t_e}^{t_0} \frac{dt}{R(t)} = \frac{1}{c}\int_0^{\varpi_e} \frac{d\varpi}{\sqrt{1-k\varpi^2}} = \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{R(t)}$$



Both red paths are
comoving distance ϖ_e long.

$$\int_{t_0}^{t_0+\Delta t_0} \frac{dt}{R(t)} = \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{R(t)}$$

$$\frac{\Delta t_e}{R(t_e)} = \frac{\Delta t_0}{R(t_0)}$$

$$\frac{R(t_0)}{R(t_e)} = \frac{\Delta t_0}{\Delta t_e}$$

$$\frac{1}{R(t_e)} = \frac{\lambda_0}{\lambda_e} = 1 + z$$

[CO 29.142]

Proper distance

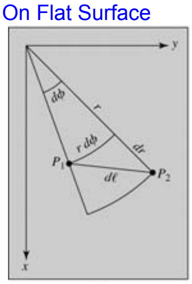
= the *current* distance to a distant object.

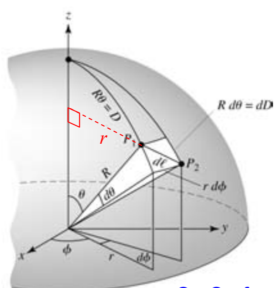
$$(ds)^2 = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + (\varpi\, d\theta)^2 + (\varpi \sin\theta\, d\phi)^2\right]$$

$dt = 0$, proper distance $d_p(t) = \mathrm{sqrt}(-ds^2)$

$$d_p(t) = R(t)\int_0^{\varpi} \frac{d\varpi'}{\sqrt{1-k\varpi'^2}}$$

On Flat Surface





On Surface of Ball

Flat: $d_{p,0} = \varpi$

Closed: $d_{p,0} = \dfrac{1}{\sqrt{k}}\sin^{-1}(\varpi\sqrt{k})$

Open: $d_{p,0} = \dfrac{1}{\sqrt{|k|}}\sinh^{-1}(\varpi\sqrt{|k|})$

The particle horizon

Horizon distance = distance a photon has traveled since $t = 0$.

$$\int_{t1}^{t2}\frac{cdt}{R(t)} = -\int_{\varpi 1}^{\varpi 2}\frac{d\varpi}{\sqrt{1-k\varpi^2}} = \int_{\varpi 2}^{\varpi 1}\frac{d\varpi}{\sqrt{1-k\varpi^2}}$$

$$d_p(t) = R(t)\int_0^{\varpi}\frac{d\varpi'}{\sqrt{1-k\varpi'^2}} = R(t)\int_{t_e}^{t_0}\frac{c\,dt'}{R(t')}$$

$$\boxed{d_h(t) = R(t)\int_0^t \frac{c\,dt'}{R(t')}}$$

Radiation dominated flat universe: $R \propto t^{1/2}$ ➔ $d_h(t) = 2ct$

Matter dominated flat universe: $R \propto t^{2/3}$ ➔ $d_h(t) = 3ct$

Matter dominated flat universe in terms of *redshift* ➔ $d_h(z) = \dfrac{2c}{H_0\sqrt{\Omega_{m,0}}}\dfrac{1}{(1+z)^{3/2}}$

Including Ω_Λ ➔ $d_h(t) = \left(\dfrac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}\sinh^{2/3}\left(\dfrac{3}{2}H_0 t\sqrt{\Omega_{\Lambda,0}}\right)\displaystyle\int_0^t \frac{c\,dt'}{\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}\sinh^{2/3}\left(\frac{3}{2}H_0 t'\sqrt{\Omega_{\Lambda,0}}\right)}$

= 14.6 Gpc (WMAP)

[29.158]

The paths of photons in terms of proper distance.

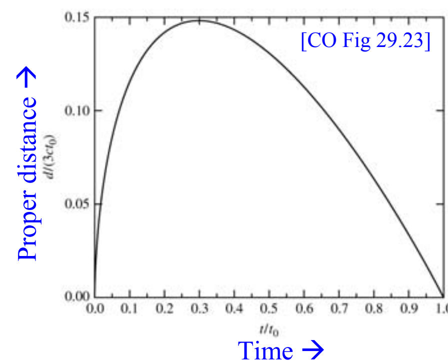
$$\cancelto{0}{(ds)^2} = (c\,dt)^2 - R^2(t)\left[\left(\frac{d\varpi}{\sqrt{1-k\varpi^2}}\right)^2 + \cancelto{0}{(\varpi\,d\theta)^2} + \cancelto{0}{(\varpi\sin\theta\,d\phi)^2}\right]$$

Matter dominated flat universe:

$$\int_0^t \frac{c\,dt'}{R(t')} = \int_{\varpi}^{\varpi_e} d\varpi'$$

$$R(t) = \left(\frac{3}{2}\right)^{\frac{2}{3}}\left(\frac{t}{t_H}\right)^{\frac{2}{3}} = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$$

$$\varpi = \varpi_e - 3ct_0\left(\frac{t}{t_0}\right)^{1/3}.$$



At $t = t_o$, $\varpi = 0$ ➔ $\varpi_e = 3ct_0$

Proper distance:

$$R(t)\varpi = d_p(t) = 3ct_0\left[\left(\frac{t}{t_0}\right)^{2/3} - \left(\frac{t}{t_0}\right)\right] \qquad \boxed{[29.165]}$$