Question 1. The R-W metric for the case $k > 0$ actually describes the equivalent of a 3D “spherical” surface embedded in a 4D space. It is easy to show this if you know the little tricks to use. The following steps will guide you through the problem. Write your answers on a separate sheet of paper, showing each intermediate result.

In Cartesian coordinates, the equation of a 1D circular “surface” of radius $R$ embedded in a 2D space is: $x^2 + y^2 = R^2$. The line element (or metric) in Cartesian coordinates in that 2D space is $dl^2 = dx^2 + dy^2$.

(a) Write down the equation of a 2D spherical surface of radius $R$ embedded in a 3D space (this is just what we usually think of as a sphere), and of the line element, both in Cartesian coordinates in that 3D space.

(b) Write down the equation of a 3D “spherical surface” of radius $R$ embedded in a 4D space, and of the line element, both in Cartesian coordinates in that 4D space. Call your dimensions $x, y, z$ and $w$.

(c) In your equation for the 4D sphere, make the replacement $r^2 = x^2 + y^2 + z^2$.

Now solve for $w^2$ in terms of $R^2$ and $r^2$, and then for $dw = function (R, r, dr)$.

(d) In your 4D line element from step (b), replace $dw$ with the results from step (c).

(e) Also replace the 3D part of the line element, $dx^2 + dy^2 + dz^2$ with its 3D equivalent in spherical coordinates $r, \theta, \phi$. Hint: if you don’t know what these are, compare the first two equations on slide 1 of the Oct. 19 lecture notes.

(f) Collect together the terms in $dr^2$ and use a little algebra. You should come up with a line element for which the coefficient of the $dr^2$ term is $1/(1 - r^2/R^2)$.

(g) Finally, make the substitutions $k^{1/2} \sigma = r/R$, $k^{1/2} d\sigma = dr/R$, and $R(t) = k^{1/2} R$. Here $R(t)$ is the scale factor and $k$ is the constant in the RW metric as used by [CO]. Write down your final answer. How does it compare with the space-like part of the RW metric for $k = +1$?

(h) So what in the world just happened? Write down 1-sentence answers to the following:

What does $R$ represent physically?

What does $r$ represent physically?

What does $k$ represent in terms of this geometry?

What new constraint allowed you to eliminate $dw$ from the 4D line element (i.e from the 4D metric)? You replaced the $w$ dimension with something, somehow, somewhere.
Question 2. Solve the Friedmann equation for the case in which the density terms $\rho_{\text{matter}}$ and $\rho_{\text{relativistic}}$ are negligible compared to the cosmological constant $\Lambda$, in a flat universe. Show your steps.

Question 3. We want to show that for the simple case $\Omega_\Lambda = \Omega_{\text{rel}} = 0$ and $\Omega_{\text{matter}} = \Omega_{m,0} = 1$, the angular diameter vs. redshift relation in [CO eqn. 29.193) becomes

$$ \frac{c\theta}{H_0 D} = \frac{1}{2} \left( \frac{(1 + z)^{3/2}}{\sqrt{1 + z - 1}} \right) $$

To do this, we need to find the correct relation between $\varpi$ and redshift $z$, and plug it into the formula on [CO pg. 1216].

On a separate sheet of paper, write down your equations at each step, so that I can see that you really worked your way through this. The word questions are just things for you to think about as you do the problem – there is no need to answer them in writing.

Start with [CO eqn. 29.138]:

$$ \int_{t_e}^{t_0} \frac{c \, dt}{R(t)} = \int_0^{\varpi_e} \frac{d\varpi}{\sqrt{1 - k \varpi^2}} $$

This is just integrating over the R-W metric for $ds = d\theta = d\phi = 0$. Why is $ds = 0$? And think carefully about why these are the correct integration limits to use. What is the correct value of $k$ to use for the parameters given at the start of this question?

But before you integrate the left-hand side, replace the integration variable $dt/R(t)$ with $\frac{dR}{R \times (dR/\, dt)}$. What are the new integration limits?

Now you need to use the solution to the Friedmann eqn. to find $dR/dt$ as a function of $R$. Try using the solution given in [CO eqn. 29.31], which is correct for this flat, matter-dominated universe. Remember that by definition $t_H = 1/H_0$.

Now replace $dR/dt$ with the correct function of $R$ and do the integrals over your modified version of eq. 29.138. Your answer should express $\varpi_e$ in terms of $c$, $t_0$, $R(t_0)$ and $R(t_e)$.

What is always the value of $R(t_0)$? What is $R(t_e)$ in terms of the redshift $z$? Plug those values into your solution. You should now have derived the $\varpi - z$ relation for a flat, matter-dominated universe.

Finally, plug your solution for $\varpi$ into the eqn. on [CO pg. 1216] and rearrange to get

$$ \frac{c \theta}{H_0 D} = \frac{1}{2} \left( \frac{(1 + z)^{3/2}}{\sqrt{1 + z - 1}} \right) $$

Your result is the same as the top curve in [CO Fig. 29.30].

Question 4. Suppose that we were still back in the radiation-dominated universe (the “radiation era”) and a photon was just arriving from the particle horizon at that time. Call the time $t_e$. Calculate the path that the photon would have taken to arrive at $\varpi = 0$ at time $t = t_e$, in terms of the proper distance $d_p(t)$ as a function of time. What is its maximum proper distance?