AST 308 Homework Set 6
Due Monday Nov 2

Question 1. The R-W metric for the case $k > 0$ actually describes the equivalent of a 3D "spherical" surface embedded in a 4D space. It is easy to show this if you know the little tricks to use. The following steps will guide you through the problem. Write your answers on a separate sheet of paper, showing each intermediate result.

In Cartesian coordinates, the equation of a 1D circular "surface" of radius $\mathcal{R}$ embedded in a 2D space is: $x^2 + y^2 = \mathcal{R}^2$. The line element (or metric) in Cartesian coordinates in that 2D space is $dl^2 = dx^2 + dy^2$.

(a) Write down the equation of a 2D spherical surface of radius $\mathcal{R}$ embedded in a 3D space (this is just what we usually think of as a sphere), and of the line element, both in Cartesian coordinates in that 3D space.

(b) Write down the equation of a 3D "spherical surface" of radius $\mathcal{R}$ embedded in a 4D space, and of the line element, both in Cartesian coordinates in that 4D space. Call your dimensions x, y, z and w:

Just in case: Here is an intermediate answer, upside down and mirror-imaged.

$$dw = -\frac{r\,dr}{w} = -\frac{r\,dr}{\left(\mathcal{R}^2 - r^2\right)^{\frac{1}{2}}}$$

(c) In your equation for the 4D sphere, make the replacement $r^2 = x^2 + y^2 + z^2$.
Now solve for w^2 in terms of $\mathcal{R}^2$ and r^2, and then for dw = function ($\mathcal{R}, r, dr$).

(d) In your 4D line element from step (b), replace dw with the results from step (c).

(e). Also replace the 3D part of the line element, $dx^2 + dy^2 + dz^2$ with its 3D equivalent in spherical coordinates r, θ, ϕ: *Hint:* if you don't know what these are, compare the first two equations on slide 1 of the Oct. 19 lecture notes.

(f) Collect together the terms in dr^2 and use a little algebra. You should come up with a line element for which the coefficient of the dr^2 term is $1/(1 - r^2/\mathcal{R}^2)$.

$$dl^2 = \left(\frac{dr^2}{1 - r^2/\mathcal{R}^2}\right) + (r\,d\theta)^2 + (r\sin\theta\, d\phi)^2$$

(g) Finally, make the substitutions $k^{½}\varpi = r/\mathcal{R}$, $k^{½}d\varpi = dr/\mathcal{R}$, and $R(t) = k^{½}\mathcal{R}$. Here $R(t)$ is the scale factor and k is the constant in the RW metric as used by [CO]. Write down your final answer. How does it compare with the space-like part of the RW metric for $k = +1$?

(h) So *what in the world* just happened? Write down 1-sentence answers to the following:

What does $\mathcal{R}$ represent physically?

What does r represent physically?

What does k represent in terms of this geometry?

What new constraint allowed you to eliminate dw from the 4D line element (*i.e* from the 4D metric)? You replaced the w dimension with something, somehow, somewhere.

Question 2. Solve the Friedmann equation for the case in which the density terms ρ_{matter} and $\rho_{relativistic}$ are negligible compared to the cosmological constant Λ, in a flat universe. Show your steps.

Question 3. We want to show that for the simple case $\Omega_\Lambda = \Omega_{\rm rel} = 0$ and $\Omega_{\rm matter} = \Omega_{\rm m,0} = 1$, the angular diameter vs. redshift relation in [CO eqn. 29.193) becomes

$$\frac{c\theta}{H_0 D} = \frac{1}{2}\frac{(1+z)^{3/2}}{\left(\sqrt{1+z}-1\right)}$$

To do this, we need to find the correct relation between ϖ and redshift z, and plug it into the formula on [CO pg. 1216].

On a separate sheet of paper, write down your equations at each step, so that I can see that you really worked your way through this. The word questions are just things for you to think about as you do the problem – there is no need to answer them in writing.

Start with [CO eqn. 29.138]: $$\int_{t_e}^{t_0} \frac{c\,dt}{R(t)} = \int_0^{\varpi_e} \frac{d\varpi}{\sqrt{1-k\varpi^2}}$$

This is just integrating over the R-W metric for $ds = d\theta = d\phi = 0$. Why is $ds = 0$? And think carefully about why these are the correct integration limits to use. What is the correct value of k to use for the parameters given at the start of this question?

But before you integrate the left-hand side, replace the integration variable $dt/R(t)$ with $\dfrac{dR}{R\times(dR/dt)}$. What are the new integration limits?

Now you need to use the solution to the Friedmann eqn. to find dR/dt as a function of R. Try using the solution given in [CO eqn. 29.31], which is correct for this flat, matter-dominated universe. Remember that by definition $t_H = 1/H_0$.

Now replace dR/dt with the correct function of R and do the integrals over your modified version of eq. 29.138. Your answer should express ϖ_e in terms of c, t_0, $R(t_0)$ and $R(t_e)$.

$$\varpi_e = \frac{2c}{H_0}\left[R(t_0)^{1/2} - R(t_e)^{1/2}\right]$$

What is *always* the value of $R(t_o)$? What is $R(t_e)$ in terms of the redshift z? Plug those values into your solution. You should now have derived the $\varpi - z$ relation for a flat, matter-dominated universe.

Finally, plug your solution for ϖ into the eqn. on [CO pg. 1216] and rearrange to get $\dfrac{c\theta}{H_0 D} = \dfrac{1}{2}\dfrac{(1+z)^{3/2}}{\left(\sqrt{1+z}-1\right)}$.

Your result is the same as the top curve in [CO Fig. 29.30].

Question 4. Suppose that we were still back in the radiation-dominated universe (the "radiation era") and a photon was just arriving from the particle horizon at that time. Call the time t_x. Calculate the path that the photon would have taken to arrive at $\varpi = 0$ at time $t = t_x$, in terms of *the proper distance* $d_p(t)$ as a function of time. What is its maximum proper distance?