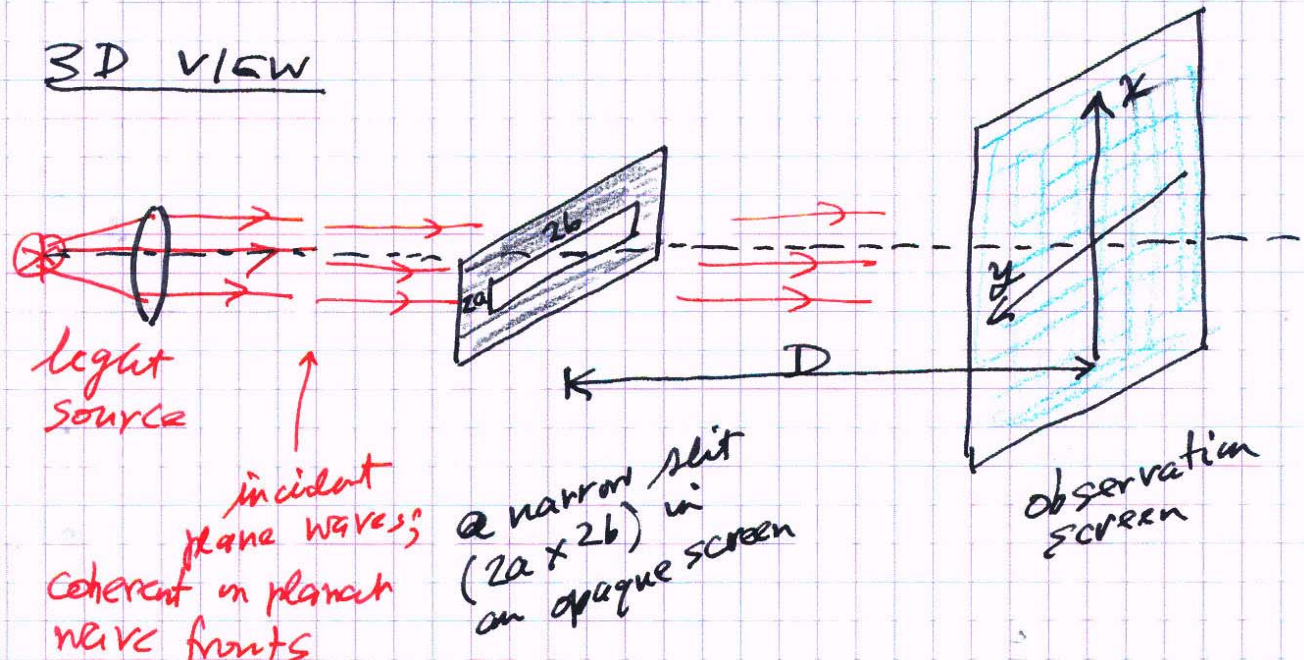
7. Diffraction

Diffraction: Bending of light, and the related interference, as light waves pass an edge or obstruction.

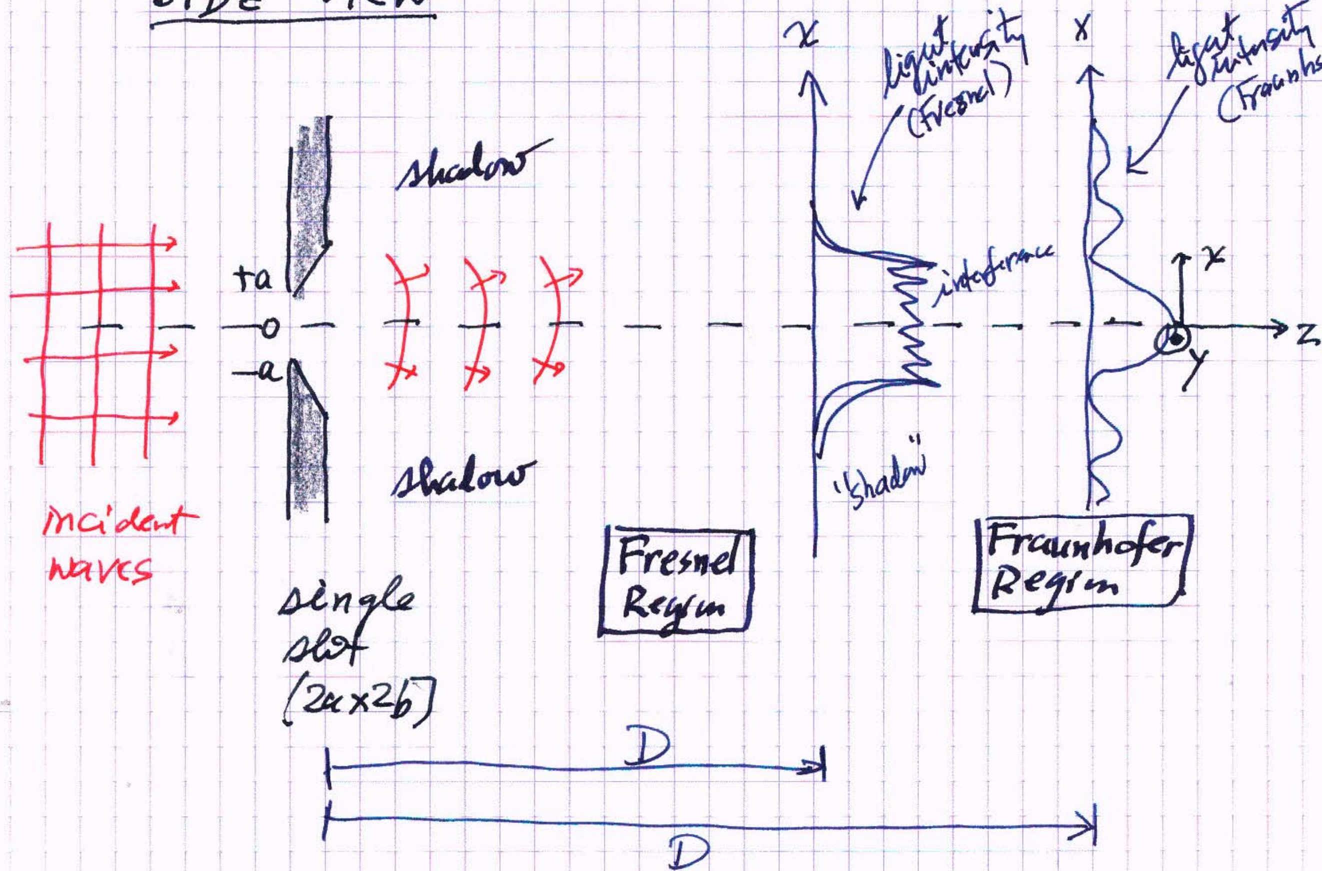
Single-slit diffraction

3D VIEW



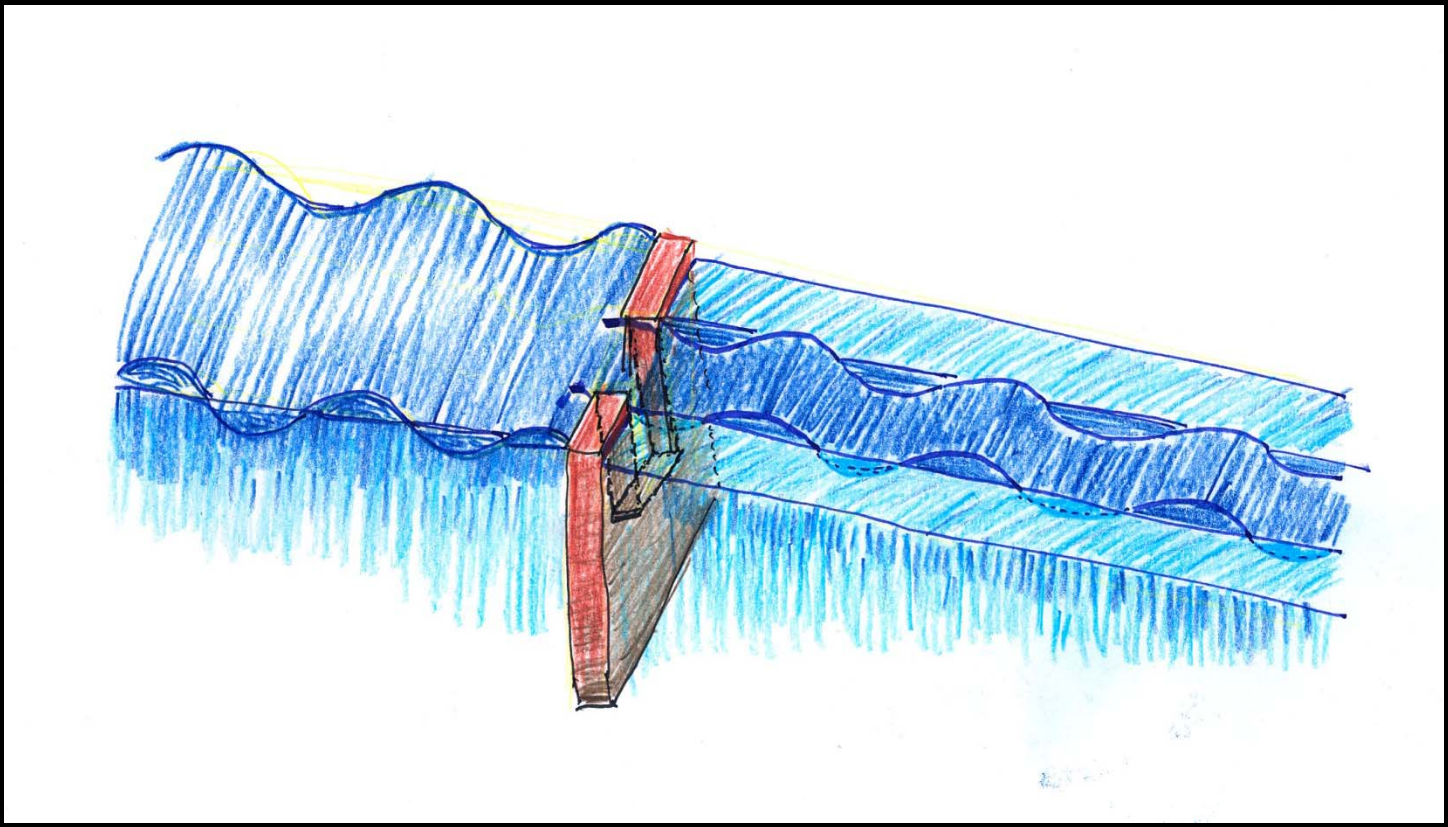
What is the light intensity $I(x, y; D)$ on the observation screen? The answer depends on D.

SIDE VIEW



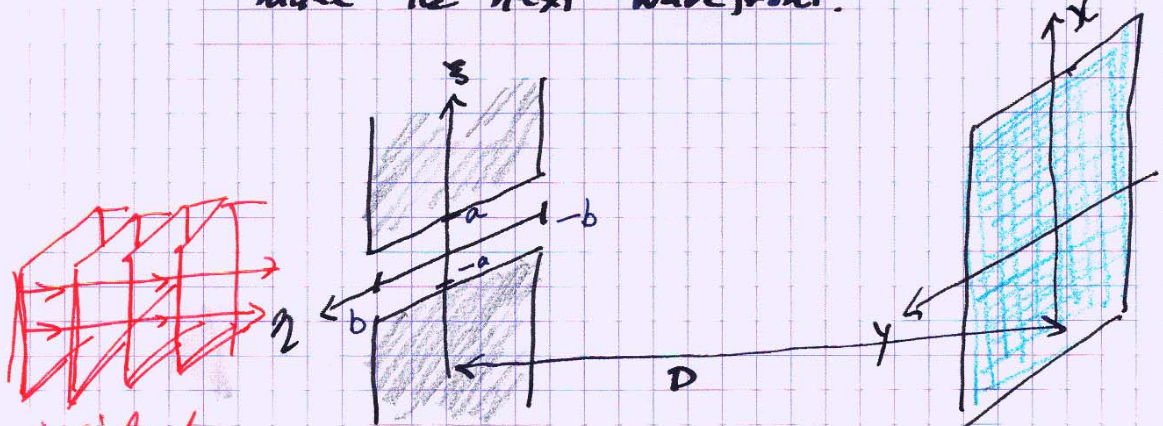
In either case, D is large compared to $\underline{a}$ or $\underline{\lambda}$. D is larger in Fraunhofer region.

Why diffraction must occur!



The medium cannot support a discontinuity.

<u>Huygens' Principle</u> Each point on a wave front acts as a source of "spherical wavelets" which propagate to make the next wavefront.



incident wave fronts

slit
coordinates
(ξ, η)

observation screen
coordinates
(x, y)

The Fresnel–Kirchoff diffraction integral

harmonic time dependence

$$\phi(x,y,D) = C \int_{\substack{\text{slit}\\\text{area}}} \frac{e^{ikr}}{r} \; \frac{1+\cos\theta}{2} \; dA \; \cdot \; e^{-i\omega t}$$

field — amplitude factor — sherical wave — "obliquity factor" ⊛ — superposition

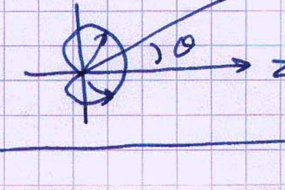
$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + D^2}$$

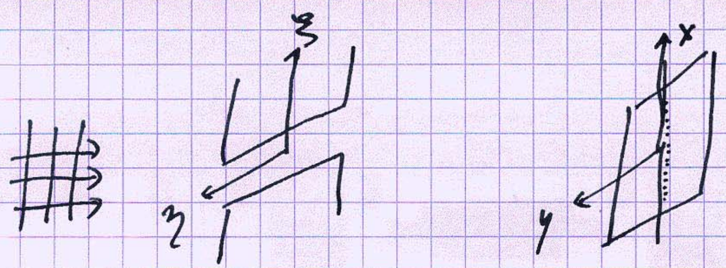
$$r \approx D + \frac{(x-\xi)^2 + (y-\eta)^2}{2D} \; {**}$$

Taylor series **
$\sqrt{D^2+\epsilon} \approx D + \frac{\epsilon}{2D}$

⊛

The wavelets are not spherical (Stokes, 1849)





Consider intensity $I(x,0,D)$ on the x axis $(y=0)$

$$\Phi(x) = C\int \frac{e^{ikr}}{r}\,\frac{1+\cos\theta}{2}\,d\xi\,d\eta \quad \cdot e^{-i\omega t}$$

Approximations for large D:

- $r = \sqrt{(x-\xi)^2+\eta^2+D^2} = D + \dfrac{(x-\xi)^2+\eta^2}{2D}$
- $\dfrac{1+\cos\theta}{2} = 1$ if θ is small
- $1/r = 1/D$
- $e^{ikr} = e^{ikD}\, e^{ik\eta^2/2D}\, e^{ik(x-\xi)^2/2D}$

$$\phi(x) = M\int_{-a}^{a} e^{ik(x-\xi)^2/2D}\,d\xi \quad e^{-i\omega t}$$

M: amplitude independent of x

$k = 2\pi/\lambda$

Let $\dfrac{\pi}{2}u^2 = \dfrac{k(x-\xi)^2}{2D}$ i.e., $u = \sqrt{\dfrac{2}{\lambda D}}\,(x-\xi)$

$du = -\sqrt{\dfrac{2}{\lambda D}}\,d\xi$

$$\phi(x) = M\sqrt{\frac{\lambda D}{2}}\int_{\sqrt{\frac{2}{\lambda D}}(x-a)}^{\sqrt{\frac{2}{\lambda D}}(x+a)} e^{i\frac{\pi}{2}u^2}\,du$$

$$\boxed{q \equiv \sqrt{\frac{2}{\lambda D}}}$$

$$\boxed{q = \sqrt{\frac{2}{\lambda D}}}$$ B3/5

$$\phi(x) = \frac{M}{q} \int_{q(x-a)}^{q(x+a)} \underbrace{e^{i\frac{\pi}{2}u^2}}_{\cos\frac{\pi}{2}u^2 + i \sin\frac{\pi}{2}u^2} du$$

(Euler)

$$\phi(x) = \frac{M}{q} \Big\{ C[q(x+a)] + iS[q(x+a)] - C[q(x-a)] - iS[q(x-a)] \Big\}$$

Fresnel Integrals

$$C(x) = \int_0^x \cos\frac{\pi}{2}u^2 \, du \qquad \text{"FresnelC"}$$

$$S(x) = \int_0^x \sin\frac{\pi}{2}u^2 \, du \qquad \text{"FresnelS"}$$

The transcendental functions are available in *Mathematica*.

Intensity on the observation screen $x, y, z = (x, 0, D)$

$$I(x) \propto |\phi(x)|^2$$

$$I(x) = I_0 \frac{R^2(x)}{R^2(0)}$$

$$R^2(x) = \big(C[q(x+a)] - C[q(x-a)]\big)^2 + \big(S[q(x+a)] - S[q(x-a)]\big)^2$$

[Link to Mathematica Notebook]

<u>Fraunhofer approximation</u> for D very large

$$\phi(x) = \frac{M}{q}\int_{q(x-a)}^{q(x+a)} e^{i\frac{\pi}{2}u^2}\,du$$

where $q = \sqrt{\frac{2}{\lambda D}}$ is small ξ

- Write $u = qx + \zeta$ when $-qa \le \zeta \le qa$
- approximate

$$u^2 = q^2x^2 + 2qx\zeta + \zeta^2 \approx q^2x^2 + 2qx\zeta$$

- and use

$$\int_{-qa}^{qa} e^{i\frac{\pi}{2}\cdot 2qx\zeta}\,d\zeta = \frac{2i\sin\left[\pi q^2 a x\right]}{i\pi q x}$$

$$\int_{-\theta}^{\theta} e^{i\alpha\phi}\,d\phi = \frac{1}{i\alpha}\left. e^{i\alpha\phi}\right|_{\phi=-\theta}^{\theta} = \frac{2i\sin\alpha\phi}{i\alpha}$$

- note $\pi q^2 a x = \frac{2\pi}{\lambda}a\frac{x}{D} \approx k a \sin\theta$

$$\frac{x}{D} = \tan\theta \approx \theta \approx \sin\theta$$

Finally,

$$I(x) = I_0\left[\frac{\sin(ka\sin\theta)}{ka\sin\theta}\right]^2$$

Fraunhofer approximation for single-slit diffraction

Khare & Swarup, Chapter 5