Administrative Announcements

• Total number of Labs/Homework sets is reduced to “10”, only 9 out 10 reports will be counted towards your final grade.

• Dr. Lai’s office hours:
  – 2-3pm Fridays @ 4238 BPS

• Mrs. Linying Lin’s office hours (homework):
  – 2-3pm Mondays @ Optics Lab/1250 BPS
Monochromatic waves

- A ‘wave’ = solution to the wave equation
- We’ll only consider monochromatic fields
  - Fourier methods are used for polychromatic light
- Electric fields are most important:
  - For monochromatic electric fields
    \[
    \mathbf{E}(\mathbf{r},t) = A(r) \cos(\omega t + \delta) \quad A : \text{real}, \delta : \text{phase}
    \]
    \[
    = \text{Re}\{\mathbf{E}(\mathbf{r})e^{i\omega t}\}
    \]
    \[
    \omega = 2\pi f
    \]
    \[
    \mathbf{E}(\mathbf{r},t) = \quad \text{Complex amplitude of the electric field vector}
    \]
    \[
    \mathbf{E}(\mathbf{r}) \quad \text{contains amplitude, and :}
    \]
    - The direction of propagation [denoted by \((\mathbf{r})\)]
    - The phase of the light (complex)
    - The polarization: ‘direction’ of \(\mathbf{E}\) (as in linear polarization)

\[
\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0
\]

\[
H, P, D, M, B \quad \text{are similarly defined} \]
Monochromatic plane waves

Plane waves have straight wave fronts

- As opposed to spherical waves, etc.

- Suppose

$$E(r) = E_0 e^{ik \cdot r}$$

$$E(r, t) = \text{Re}\{E(r)e^{-i\omega t}\}$$

$$= \text{Re}\{E_0 e^{i(k \cdot r - \omega t)}\}$$

- $E_0$ still contains: amplitude, polarization, phase

- Direction of propagation given by wavevector:

$$k = (k_x, k_y, k_z) \text{ where } |k| = 2\pi/\lambda = \omega/c$$

- Can also define

$$E = (E_x, E_y, E_z)$$

- Plane wave propagating in z-direction

$$E(z, t) = \text{Re}\{E_0 e^{i(kz - \omega t)}\} = \frac{1}{2} \{E_0 e^{i(kz - \omega t)} + E_0^* e^{-i(kz - \omega t)}\}$$
In isotropic media (e.g. free space, amorphous glass, etc.)

\[ \mathbf{k} \cdot \mathbf{E} = 0 \]
i.e. \( \mathbf{k} \perp \mathbf{E} \)

More generally, \( \mathbf{k} \cdot \mathbf{D} = 0 \)

(reminder: in anisotropic media, e.g. crystals, one could have \( \mathbf{E} \) not parallel to \( \mathbf{D} \))
Wave equations

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{where} \quad \mathbf{E} = \hat{x}E_0 \ e^{i(k \cdot r - \omega t)} \]

\[ \Rightarrow \nabla \times \equiv i\mathbf{k} \times \quad \text{and} \quad \frac{\partial}{\partial t} \equiv -i\omega \]

\[ \Rightarrow \mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E} \]

Vectors \( \mathbf{k}, \mathbf{E}, \mathbf{B} \) form a right-handed triad.

**Note:** free space or isotropic media only
The Poynting vector

\[ S = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = c^2 \varepsilon_0 \mathbf{E} \times \mathbf{B} \]

so in free space

\[ S \parallel k \]

S has units of W/m²
so it represents
energy flux (energy per unit time & unit area)
The Poynting vector: part II

\[ \mathbf{S} = c^2 \varepsilon_0 \mathbf{E} \times \mathbf{B} \]

\[ \mathbf{B} = \frac{k}{\omega} \times \mathbf{E} \Rightarrow \|\mathbf{B}\| = \frac{k}{\omega} \|\mathbf{E}\| = \frac{1}{c} \|\mathbf{E}\| \Rightarrow \|\mathbf{S}\| = c \varepsilon_0 \|\mathbf{E}\|^2 \]

For example, sinusoidal field propagating along \( z \)

\[ \mathbf{E} = \hat{x} E_0 \cos(kz - \omega t) \Rightarrow \|\mathbf{S}\| = c \varepsilon_0 E_0^2 \cos^2(kz - \omega t) \]

Recall: for visible light, \( \omega \sim 10^{14} - 10^{15} \text{Hz} \)
Recall: for visible light, $\omega \sim 10^{14} - 10^{15}$ Hz

So any instrument will record the *average* incident energy flux

$$\langle \| \mathbf{S} \| \rangle = \frac{1}{T} \int_{t}^{t+T} \| \mathbf{S} \| \, dt \quad \text{where } T \text{ is the period } (T=\lambda/c)$$

$\langle \| \mathbf{S} \| \rangle$ is called the *irradiance*, aka *intensity* of the optical field (units: W/m$^2$)
For example: sinusoidal electric field,

\[ \mathbf{E} = \hat{x} E_0 \cos(kz - \omega t) \implies \|S\| = c \varepsilon_0 E_0^2 \cos^2(kz - \omega t) \]

Then, at constant \( z \):

\[
\left\langle \cos^2(kz - \omega t) \right\rangle = \int_{t}^{t+T} \cos^2(kz - \omega t) dt = \frac{1}{2}
\]

\[
\left\langle \|S\| \right\rangle = \frac{1}{2} c \varepsilon_0 E_0^2
\]
Intensity of Light

Summary (free space or isotropic media)

\[ S = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}; \quad \|S\| = c\varepsilon_0 \|\mathbf{E}\|^2 \]  
Poynting vector

\[ \langle \|S\| \rangle = \frac{1}{T} \int_{t}^{t+T} \|S\| \, dt \]  
Irradiance (or intensity)

- Poynting vector describes flows of E-M power
- Power flow is directed along this vector
  - Usually parallel to \( \mathbf{k} \)
- Intensity is equal to the magnitude of the time averaged Poyning vector: \( I = \langle S \rangle \)

\[ \langle S \rangle = I \equiv \langle \mathbf{E}(t) \times \mathbf{H}(t) \rangle = \frac{c\varepsilon_0}{2} E^2 = \frac{c\varepsilon_0}{2} \left( E_x^2 + E_y^2 \right) \]
\[ c\varepsilon_0 \approx 2.654 \times 10^{-3} \, A / V \]

**example**

\[ E = 1 \, V \, / \, m \]
\[ I = ? \, W \, / \, m^2 \]

\[ \hbar \omega \, [eV] = \frac{1239.85}{\lambda \, [nm]} \]
\[ \hbar = 1.05457266 \times 10^{-34} \, Js \]
Maxwell’s equations in a medium (source-free)

The induced polarization, \( P \), contains the effect of the medium:

\[
\nabla \cdot \vec{E} = 0 \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\
\nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \frac{\partial \vec{P}}{\partial t}
\]

\( \vec{B} = \mu_0 \vec{H} \)

\( \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \)

The polarization is proportional to the field:

\( \vec{P} = \varepsilon_0 \chi \vec{E} \)

This has the effect of simply changing the dielectric constant (refractive index \( n \)):

\( \varepsilon = \varepsilon_0 (1 + \chi) = n^2 \)
Wave equations in a medium

The induced polarization in Maxwell’s Equations yields another term in the wave equation:

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{dt^2}$$

This is the **Inhomogeneous Wave Equation**.

The polarization is the driving term for a new solution to this equation.
Reflection and Transmission @ dielectric interface
Reflection and Transmission

\[
\begin{align*}
\mathbf{r}_\perp &= \left( \frac{E_{0r}}{E_{0i}} \right)_\perp = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} \\
\mathbf{t}_\perp &= \left( \frac{E_{0t}}{E_{0i}} \right)_\perp = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_t \cos \theta_t} \\
\mathbf{r}_\parallel &= \left( \frac{E_{0r}}{E_{0i}} \right)_\parallel = \frac{n_t \cos \theta_i - n_i \cos \theta_t}{n_t \cos \theta_i + n_i \cos \theta_t} \\
\mathbf{t}_\parallel &= \left( \frac{E_{0t}}{E_{0i}} \right)_\parallel = \frac{2n_i \cos \theta_t}{n_t \cos \theta_i + n_i \cos \theta_t}
\end{align*}
\]