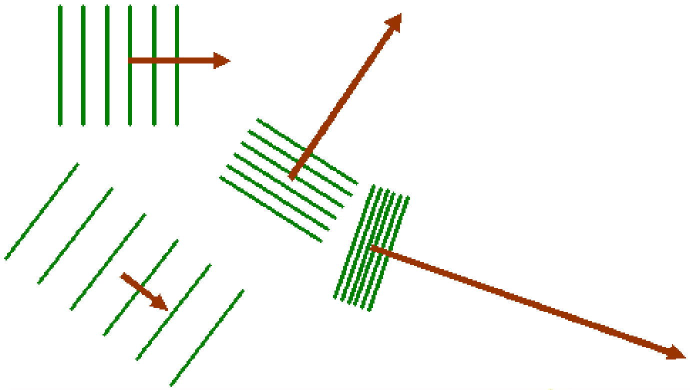
Monochromatic plane waves



Plane waves have straight wave fronts

– As opposed to spherical waves, etc.

– Suppose

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$$

$\Rightarrow$

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathrm{Re}\{\mathbf{E}(\mathbf{r})e^{-i\omega t}\}\\ &= \mathrm{Re}\{\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}\}\\ &= \mathrm{Re}\{\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\}\end{aligned}$$

– $\mathbf{E}_0$ still contains: amplitude, polarization, phase

– Direction of propagation given by wavevector:

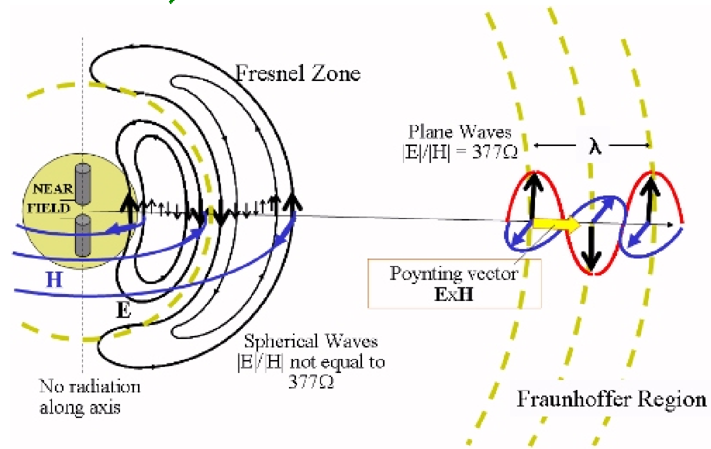
$\mathbf{k} = (k_x, k_y, k_z)$ where $|\mathbf{k}| = 2\pi/\lambda = \omega/c$

– Can also define

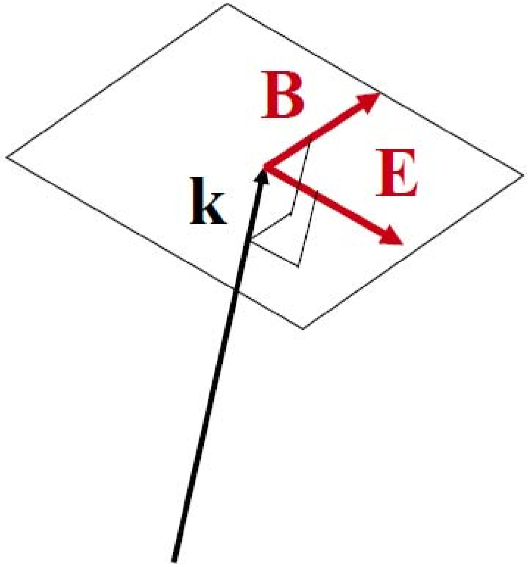
$\mathbf{E} = (E_x, E_y, E_z)$

– Plane wave propagating in z-direction

$$\mathbf{E}(z,t) = \mathrm{Re}\{\mathbf{E}_0 e^{i(kz-\omega t)}\} = \tfrac{1}{2}\{\mathbf{E}_0 e^{i(kz-\omega t)} + \mathbf{E}_0^* e^{-i(kz-\omega t)}\}$$



Wave equations



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{where} \quad \mathbf{E} = \hat{\mathbf{x}} E_0 \, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

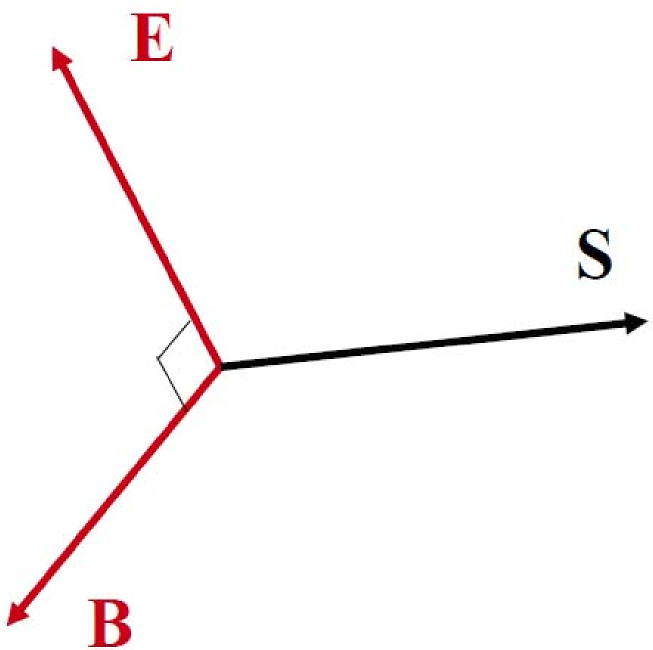
$$\Rightarrow \nabla \times \equiv i\mathbf{k} \times \quad \text{and} \quad \frac{\partial}{\partial t} \equiv -i\omega$$

$$\Rightarrow \mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}$$

Vectors **k**, **E**, **B** form a right-handed triad.

Note: free space or isotropic media only

The Poynting vector



$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = c^2\varepsilon_0\mathbf{E}\times\mathbf{B}$$

so in free space

$$\mathbf{S} \parallel \mathbf{k}$$

S has units of W/m^2
so it represents
energy flux (energy per
unit time & unit area)

Intensity of Light

Summary (free space or isotropic media)

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}; \quad \|\mathbf{S}\| = c\varepsilon_0 \|\mathbf{E}\|^2$$ Poynting vector

$$\langle \|\mathbf{S}\| \rangle = \frac{1}{T}\int_t^{t+T} \|\mathbf{S}\| \mathrm{d}t$$ Irradiance (or intensity)

- Poynting vector describes flows of E-M power
- Power flow is directed along this vector
  - Usually parallel to **k**
- Intensity is equal to the magnitude of the time averaged Poyning vector: I=<**S**>

$$|\langle \mathbf{S} \rangle| = I \equiv |\langle \mathbf{E}(t) \times \mathbf{H}(t) \rangle| = \frac{c\varepsilon_0}{2} E^2 = \frac{c\varepsilon_0}{2}\left(E_x{}^2 + E_y{}^2\right)$$

$$c\varepsilon_0 \approx 2.654 \times 10^{-3} A/V$$

example

$$E = 1V/m$$

$$I = ?\, W/m^2$$

$$\hbar\omega[eV] = \frac{1239.85}{\lambda[nm]}$$

$$\hbar = 1.05457266 \times 10^{-34}\, Js$$

The Poynting vector : part II

$$\mathbf{S} = c^2 \varepsilon_0 \mathbf{E} \times \mathbf{B}$$

$$\mathbf{B} = \frac{\mathbf{k}}{\omega} \times \mathbf{E} \Rightarrow \|\mathbf{B}\| = \frac{k}{\omega}\|\mathbf{E}\| = \frac{1}{c}\|\mathbf{E}\| \Rightarrow \|\mathbf{S}\| = c\varepsilon_0 \|\mathbf{E}\|^2$$

For example, sinusoidal field propagating along z

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \cos(kz - \omega t) \Rightarrow \|\mathbf{S}\| = c\varepsilon_0 E_0^2 \cos^2(kz - \omega t)$$

Recall: for visible light, $\omega \sim 10^{14}$-10^{15}Hz

The Poynting vector: part III

Recall: for visible light, $\omega \sim 10^{14}$-10^{15}Hz

So any instrument will record the
average incident energy flux

$$\langle \|\mathbf{S}\| \rangle = \frac{1}{T}\int_{t}^{t+T} \|\mathbf{S}\| \mathrm{d}t$$

where T is the period (T=λ/c)

$\langle \|\mathbf{S}\| \rangle$ is called the *irradiance*, aka *intensity* of the optical field (units: W/m^2)

For example: sinusoidal electric field,

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \cos(kz - \omega t) \Rightarrow \|\mathbf{S}\| = c\varepsilon_0 E_0^2 \cos^2(kz - \omega t)$$

Then, at constant z:

$$\left\langle \cos^2(kz - \omega t) \right\rangle = \int_t^{t+T} \cos^2(kz - \omega t)\mathrm{d}t = \frac{1}{2}$$

$$\left\langle \|\mathbf{S}\| \right\rangle = \frac{1}{2} c\varepsilon_0 E_0^2$$

Intensity of Light

Summary (free space or isotropic media)

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}; \quad \|\mathbf{S}\| = c\varepsilon_0 \|\mathbf{E}\|^2$$ Poynting vector

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$$c\varepsilon_0 \approx 2.654 \times 10^{-3} A / V$$

example $E = 1V / m$

$$I = ?\, W / m^2$$

$$\hbar\omega[eV] = \frac{1239.85}{\lambda[nm]}$$

$$\hbar = 1.05457266 \times 10^{-34} Js$$

Maxwell's equations in a medium (source-free)

The induced polarization, P, contains the effect of the medium:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \qquad \vec{B}=\mu\vec{H}$$

$$\vec{\nabla}\cdot\vec{H}=0 \qquad \vec{\nabla}\times\vec{H}=\varepsilon_0\frac{\partial\vec{E}}{\partial t}+\frac{\partial\vec{P}}{\partial t}=\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{D}=\varepsilon_0\vec{E}+\vec{P}=\varepsilon\vec{E}$$

$$\vec{P}=\varepsilon_0\chi\vec{E}$$

The polarization is proportional to the field:

$$\vec{P}=\varepsilon_0\chi\vec{E}$$

This has the effect of simply changing the dielectric constant (refractive index n):

$$\varepsilon=\varepsilon_0\left(1+\chi\right)=n^2$$

Wave equations in a medium

The induced polarization in Maxwell's Equations yields another term in the wave equation:

$$\frac{\partial^2 E}{\partial z^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \qquad \frac{\partial^2 E}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0$$

This is the **Inhomogeneous Wave Equation**.

The polarization is the driving term for a new solution to this equation.

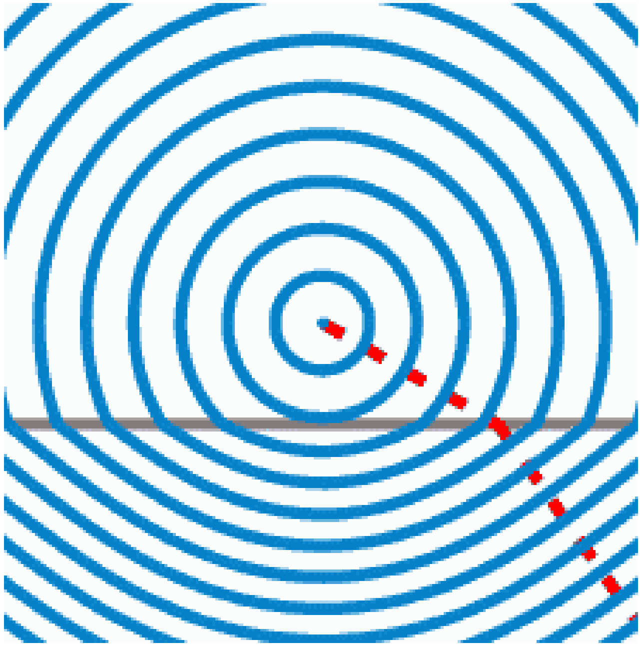
$$\frac{\partial^2 E}{\partial z^2} - \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \qquad \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Homogeneous (Vacuum) Wave Equation

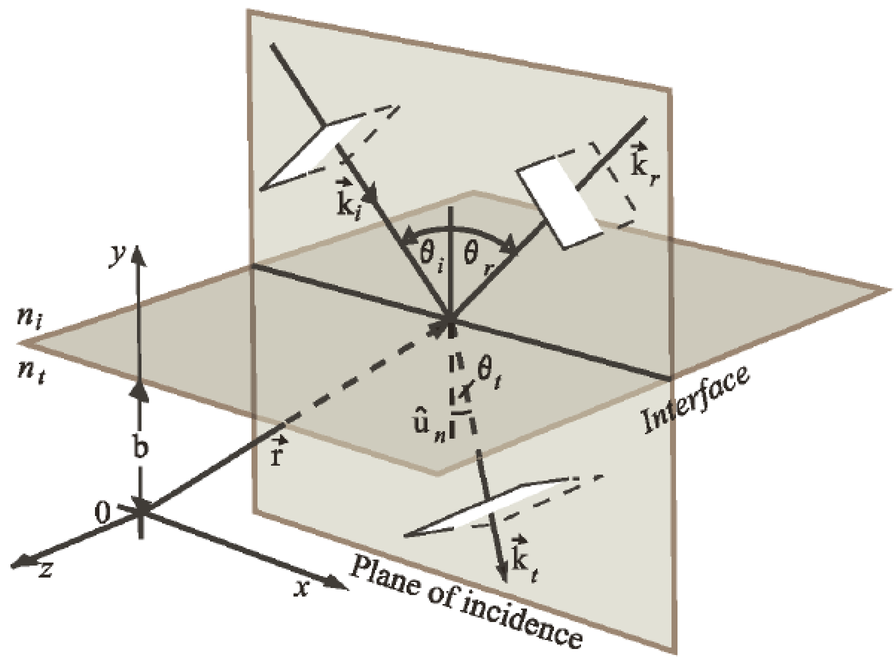
$$\begin{aligned} \mathbf{E}(z,t) &= \mathrm{Re}\{\mathbf{E}_0 e^{i(kz-\omega t)}\} \\ &= \tfrac{1}{2}\{\mathbf{E}_0 e^{i(kz-\omega t)} + \mathbf{E}_0^* e^{-i(kz-\omega t)}\} \\ &= |\mathbf{E}_0| \cos(kz-\omega t) \end{aligned}$$

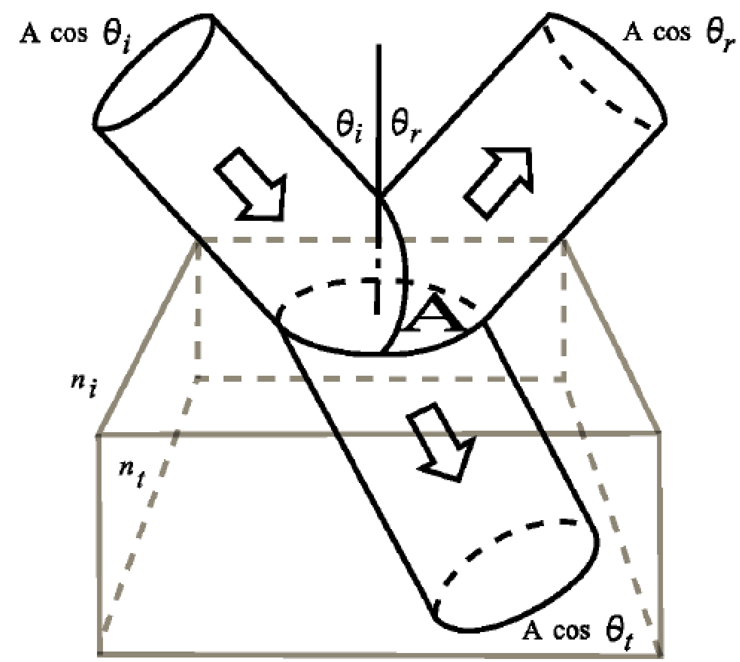
$$\frac{c}{v} = ?$$

Wave Picture

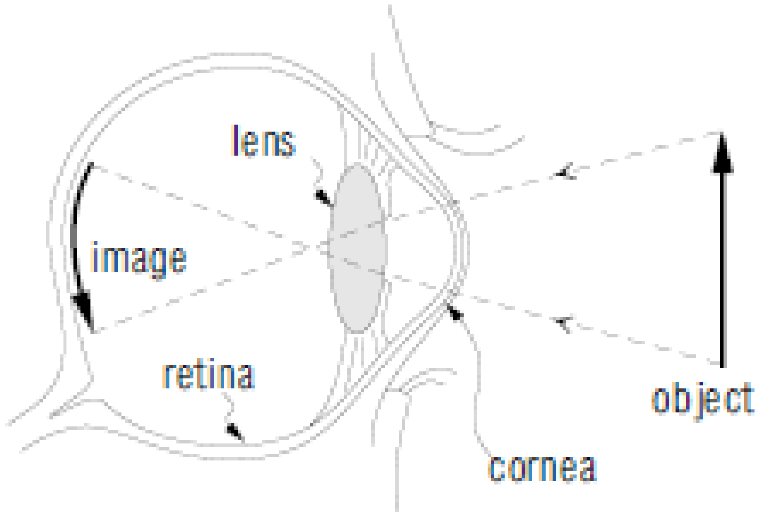


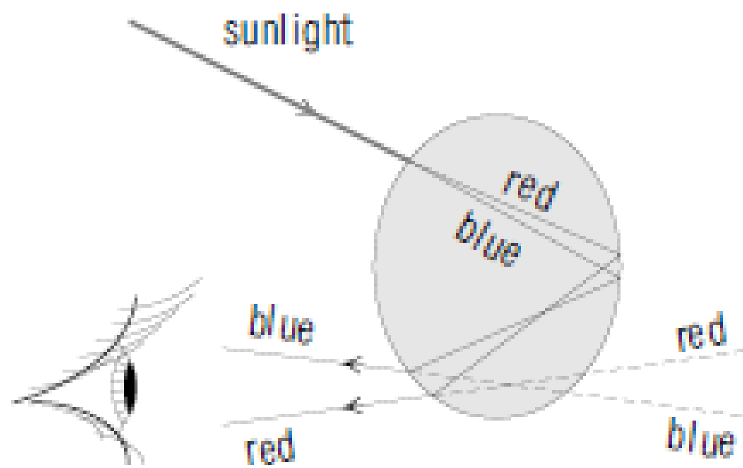
Reflection and Transmission @ dielectric interface

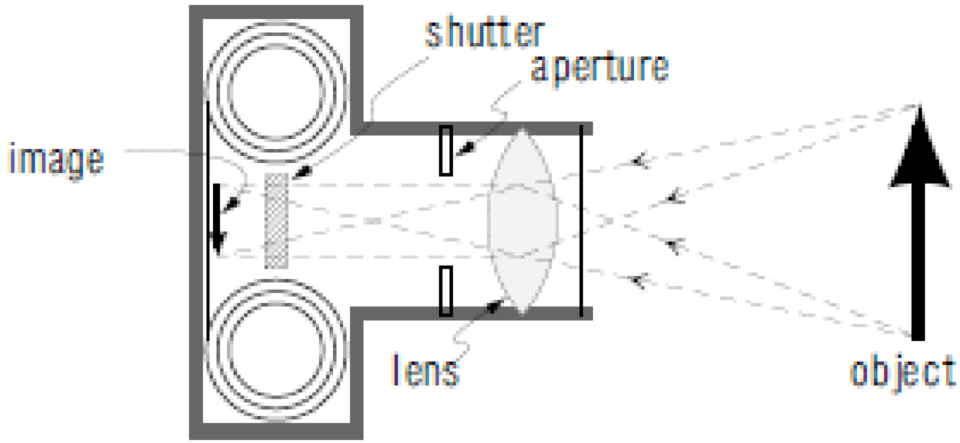




Light as Rays: Snell's Law





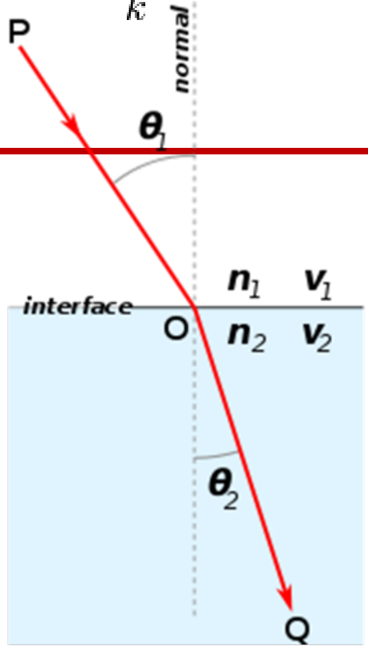


Snell's Law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \ .$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2}$$



1. Fermat's principle, which states that the light travels the path which takes the least time. By taking the derivative of the optical path length, the stationary point is found giving the path taken by the light (though it should be noted that the result does not show light taking the least time path, but rather one that is stationary with respect to small variations as there are cases where light actually takes the greatest time path, as in a spherical mirror).

- In a classic analogy, the area of lower refractive index is replaced by a beach, the area of higher refractive index by the sea, and the fastest way for a rescuer on the beach to get to a drowning person in the sea is to run along a path that follows Snell's law.

2. Snell's law can be derived using interference of all possible paths of light wave from source to observer—it results in destructive interference everywhere except extrema of phase (where interference is constructive)—which become actual paths.

3. Application of the general boundary conditions of Maxwell equations for electromagnetic radiation.

4. . Conservation of momentum based on translation symmetry considerations

Derive Snell's Law by Translation Symmetry

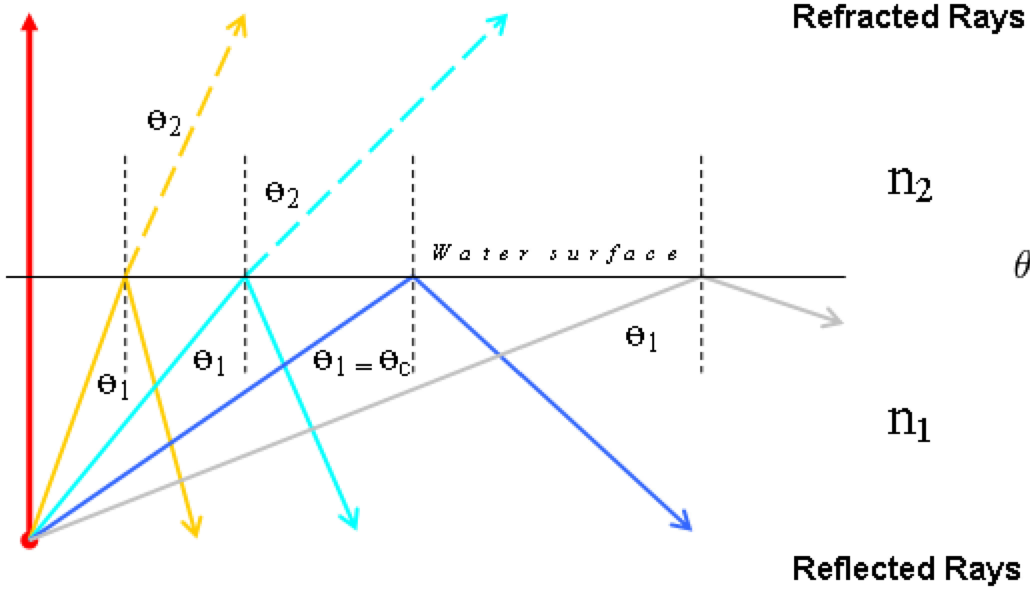
A homogeneous surface perpendicular to say the z direction can not change the transverse momentum. Since the propagation vector is proportional to the photon's momentum, the transverse propagation direction $(k_x, k_y, 0)$ must remain the same in both regions. Assuming without loss of generality a plane of incidence in the z,x plane $k_{xRegion1} = k_{xRregion2}$. Using the well known dependence of the wave number on the refractive index of the medium, we derive Snell's law immediately.

$$k_{xRegion1} = k_{xRregion2}$$
$$n_1 k_0 \sin\theta_1 = n_2 k_0 \sin\theta_2$$
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where is the wavenumber in vacuum. Note that no surface is truly homogeneous, in the least at the atomic scale. Yet full translational symmetry is an excellent approximation whenever the region is homogeneous on the scale of the light wavelength.

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega}{c}$$

Total Internal Reflectation



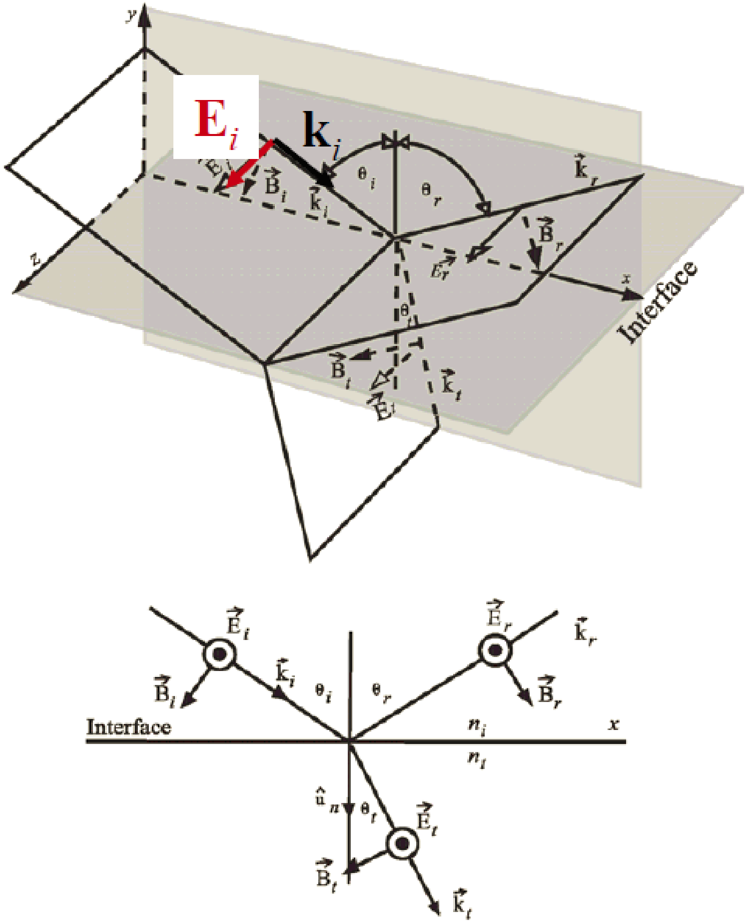
$$\theta_{\mathrm{crit}} = \arcsin\left(\frac{n_2}{n_1}\sin\theta_2\right) = \arcsin\frac{n_2}{n_1} = 48.6^\circ.$$

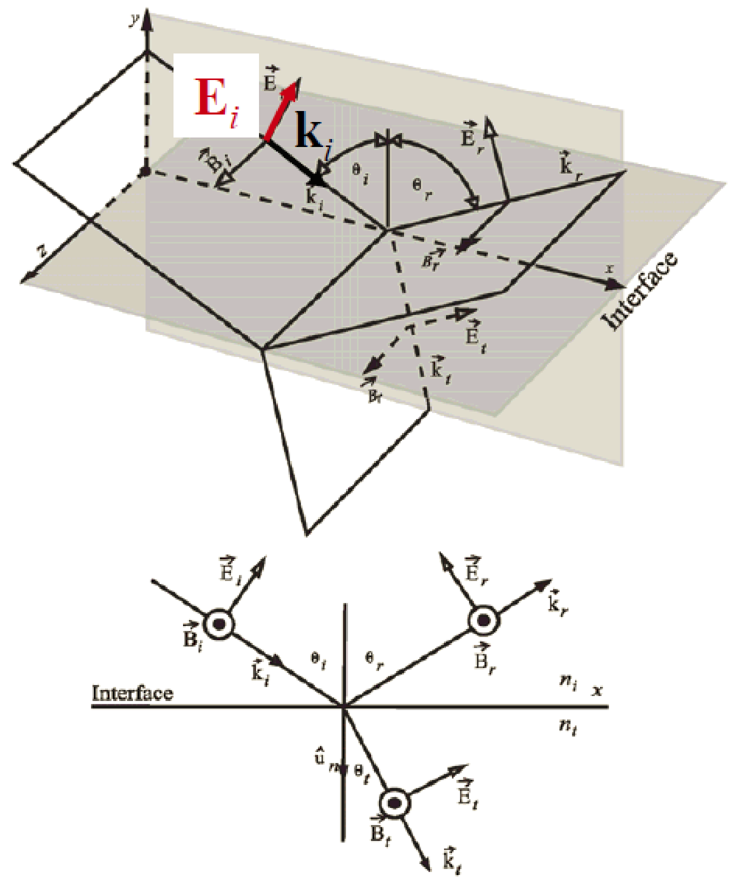
When light travels from a medium with a higher refractive index to one with a lower refractive index, Snell's law seems to require in some cases (whenever the angle of incidence is large enough) that the sine of the angle of refraction be greater than one. This of course is impossible, and the light in such cases is completely reflected by the boundary, a phenomenon known as total internal reflection. The largest possible angle of incidence which still results in a refracted ray is called the **critical angle**; in this case the refracted ray travels along the boundary between the two media.

For example, consider a ray of light moving from water to air with an angle of incidence of 50°. The refractive indices of water and air are approximately 1.333 and 1, respectively, so Snell's law gives us the relation $\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = 1.333 \cdot 0.766 = 1.021,$

which is impossible to satisfy. The critical angle θ_{crit} is the value of θ_1 for which θ_2 equals 90°:

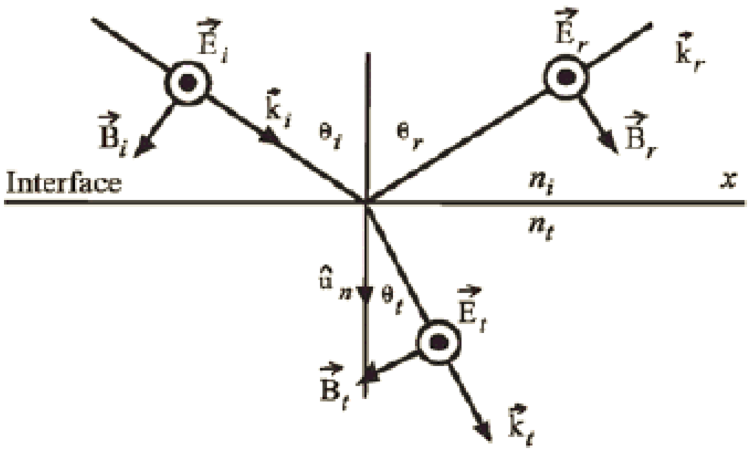
Beyond Snell's Law: Polarization?





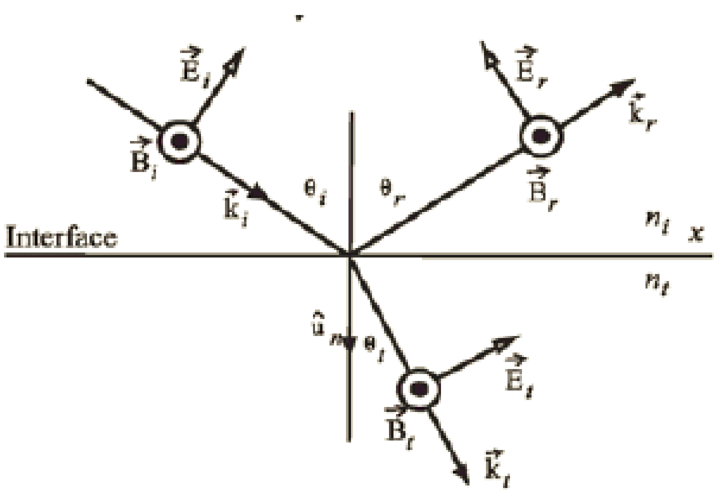
Reflection and Transmission (Fresnel's equations)

Can be deduced from the application of boundary conditions of EM waves.



$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$



$$r_\parallel = \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

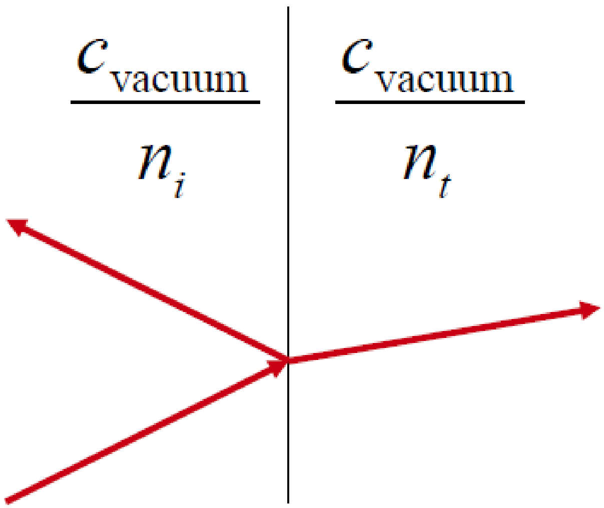
$$t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

Reflection and Transmission of Energy @ dielectric interfaces

Recall Poynting vector definition:

$$\|\mathbf{S}\| = c\,\varepsilon_0 \|\mathbf{E}\|^2$$

different on the two sides of the interface

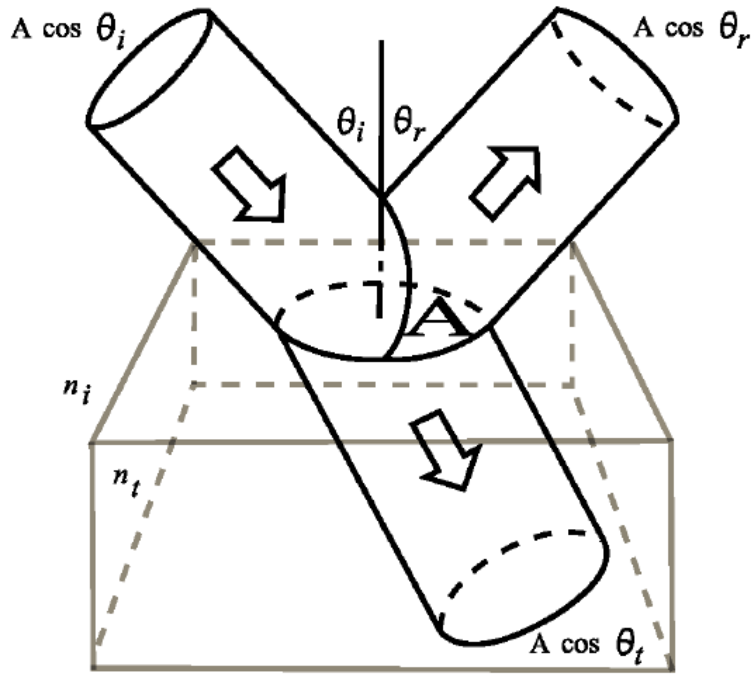


$$R = \left(\frac{E_{0r}}{E_{0i}}\right)^2 = r^2$$

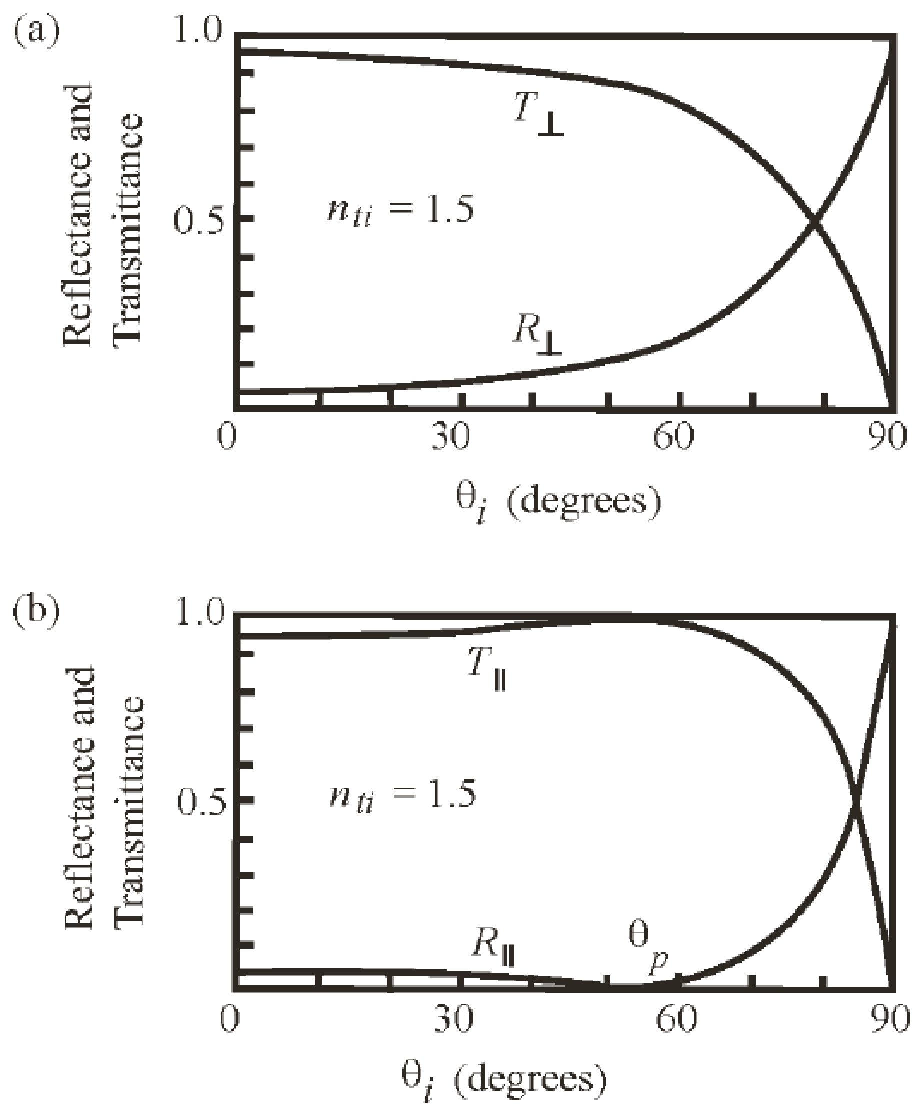
$$T = \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\left(\frac{E_{0t}}{E_{0i}}\right)^2 = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} t^2$$

Energy Conservation

$$R + T = 1, \ \text{i.e.} \quad r^2 + \frac{n_t \cos \theta_t}{n_i \cos \theta_i} t^2 = 1$$



Reflectance and Transmittance @ dielectric interfaces



Normal Incidence

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$r_\parallel = \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t}$$

$$t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t}$$

Note: independent of polarization

$\theta_i = 0$ and $\theta_t = 0$ $\Longrightarrow$

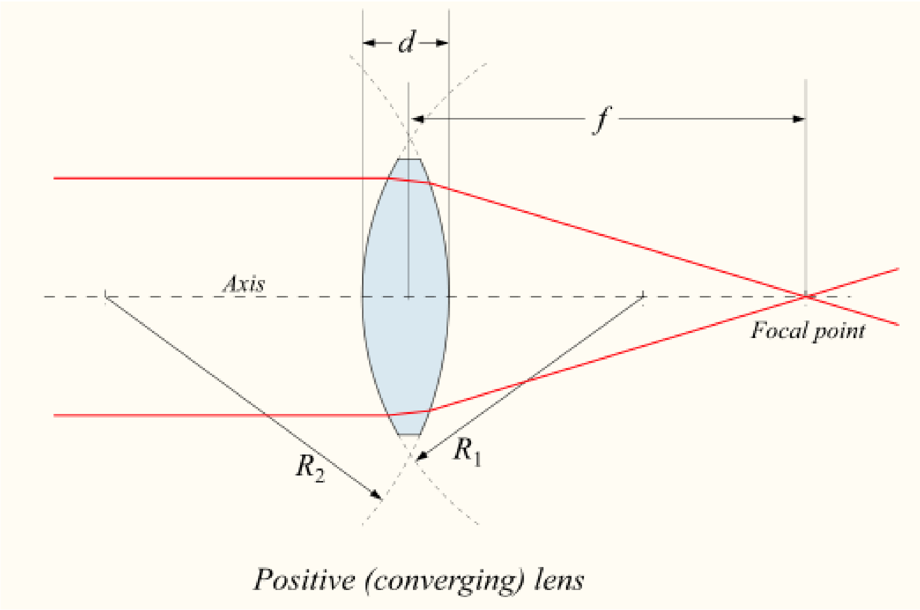
$$r_\perp = r_\parallel = \frac{n_t - n_i}{n_t + n_i}$$

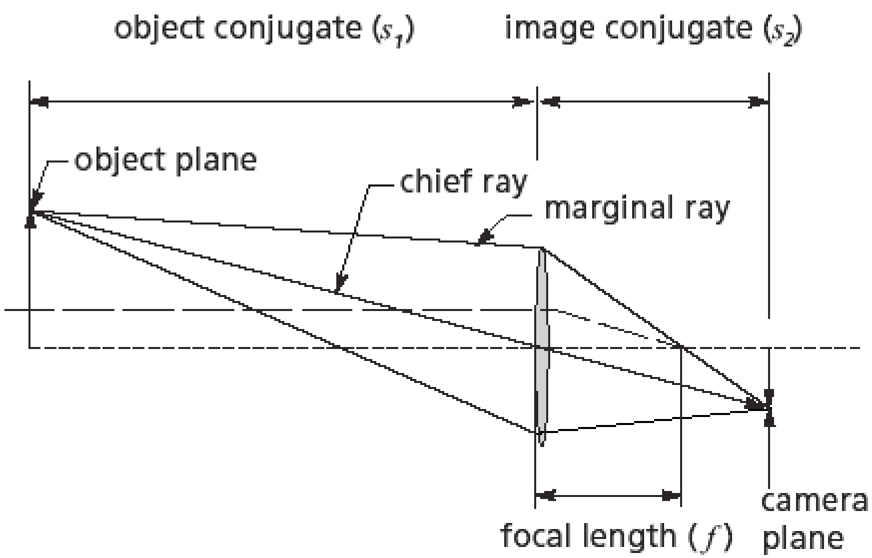
$$t_\perp = t_\parallel = \frac{2n_i}{n_t + n_i}$$

$$R_\perp = R_\parallel = \left(\frac{n_t - n_i}{n_t + n_i}\right)^2$$

$$T_\perp = T_\parallel = \frac{4n_t n_i}{(n_t + n_i)^2}$$

Thin Lens



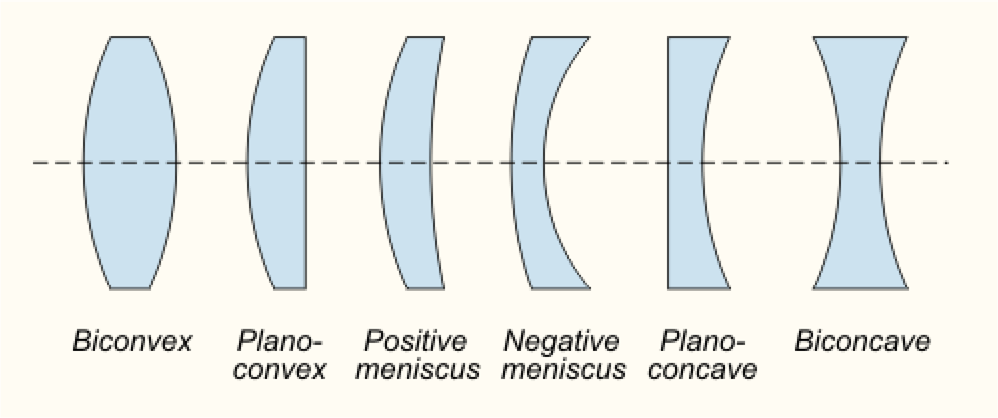


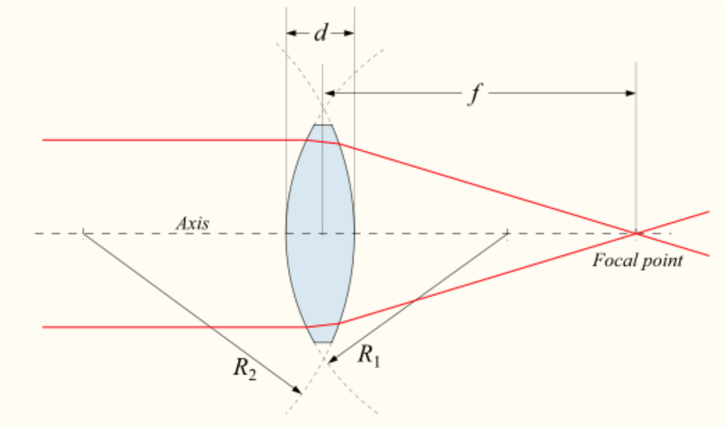
$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right],$$

$$\frac{1}{f} = \frac{1}{S_1} + \frac{1}{S_2}$$

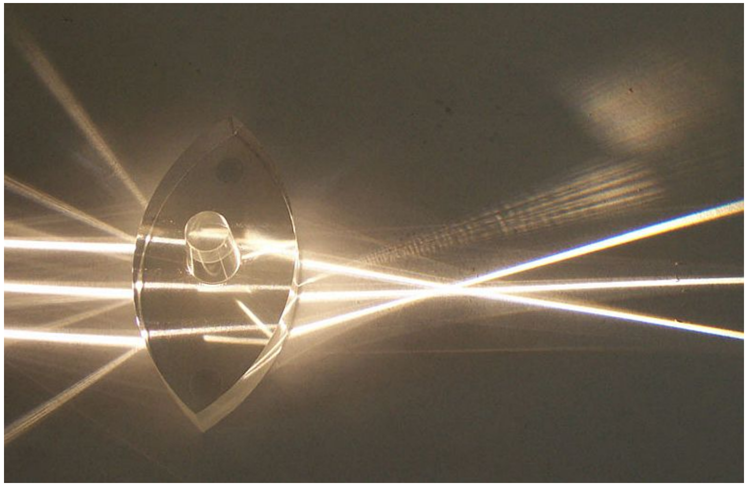
$$\frac{1}{f} \approx (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right].$$

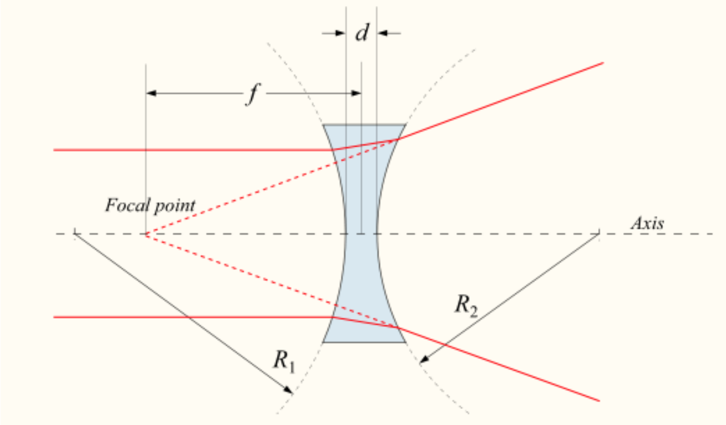
$$M = \frac{S_2}{S_1}$$





Positive (converging) lens





Negative (diverging) lens