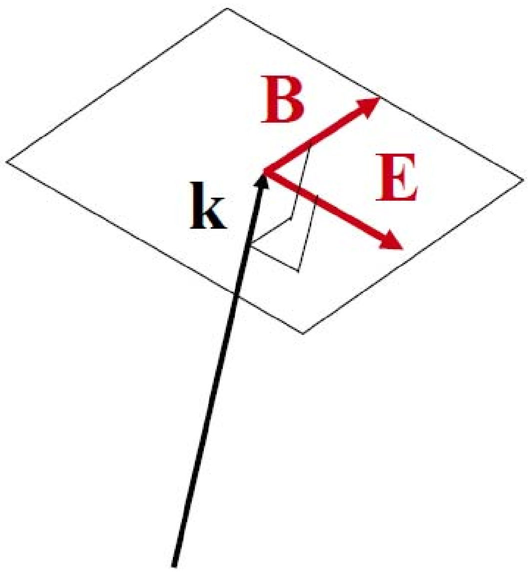
Wave equations



$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{where} \quad \mathbf{E} = \hat{\mathbf{x}} E_0 \, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$$\Rightarrow \nabla \times \equiv i\mathbf{k}\times \quad \text{and} \quad \frac{\partial}{\partial t} \equiv -i\omega$$

$$\Rightarrow \mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}$$

Vectors **k**, **E**, **B** form a right-handed triad.

Note: free space or isotropic media only

Intensity of Light

Summary (free space or isotropic media)

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}; \quad \|\mathbf{S}\| = c\varepsilon_0\|\mathbf{E}\|^2$$ Poynting vector

$$\langle\|\mathbf{S}\|\rangle = \frac{1}{T}\int_t^{t+T}\|\mathbf{S}\|\mathrm{d}t$$ Irradiance (or intensity)

- Poynting vector describes flows of E-M power
- Power flow is directed along this vector
  - Usually parallel to **k**
- Intensity is equal to the magnitude of the time averaged Poyning vector: I=<**S**>

$$|\langle\mathbf{S}\rangle| = I \equiv |\langle\mathbf{E}(t)\times\mathbf{H}(t)\rangle| = \frac{c\varepsilon_0}{2}E^2 = \frac{c\varepsilon_0}{2}\left(E_x^{\;2} + E_y^{\;2}\right)$$

$$c\varepsilon_0 \approx 2.654\times10^{-3}\,A/V$$

example

$$E = 1V/m$$

$$I = ?\,W/m^2$$

$$\hbar\omega[eV] = \frac{1239.85}{\lambda[nm]}$$

$$\hbar = 1.05457266\times10^{-34}\,Js$$

Wave equations in a medium

The induced polarization in Maxwell's Equations yields another term in the wave equation:

$$\frac{\partial^2 E}{\partial z^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \qquad \frac{\partial^2 E}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0$$

This is the **Inhomogeneous Wave Equation**.

The polarization is the driving term for a new solution to this equation.

$$\frac{\partial^2 E}{\partial z^2} - \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \qquad \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Homogeneous (Vacuum) Wave Equation

$$\begin{aligned}\mathbf{E}(z,t) &= \mathrm{Re}\{\mathbf{E}_0 e^{i(kz-\omega t)}\} \\ &= \tfrac{1}{2}\{\mathbf{E}_0 e^{i(kz-\omega t)} + \mathbf{E}_0^* e^{-i(kz-\omega t)}\} \\ &= |\mathbf{E}_0| \cos(kz-\omega t)\end{aligned}$$

$$n^2 = \frac{c^2}{v^2} = \frac{\mu\varepsilon}{\mu_0\varepsilon_0} \qquad \frac{c}{v} = n$$

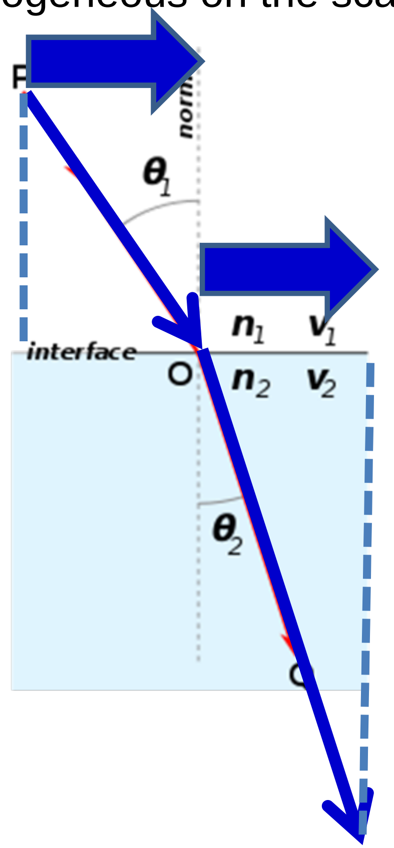
Derive Snell's Law by Translation Symmetry

$k_{xRegion1} = k_{xRregion2}$
$n_1 k_0 \sin\theta_1 = n_2 k_0 \sin\theta_2$
$n_1 \sin\theta_1 = n_2 \sin\theta_2$

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{\omega}{c}$$

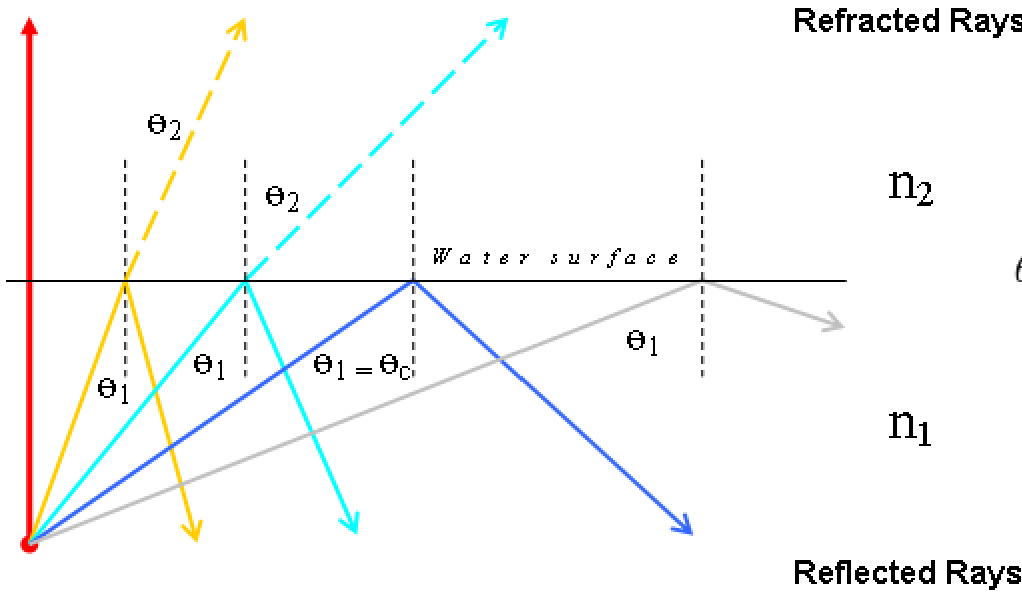
where   is the wavenumber in vacuum. Note that no surface is truly homogeneous, in the least at the atomic scale. Yet full translational symmetry is an excellent approximation whenever the region is homogeneous on the scale of the light wavelength.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \; .$$



Fermat's principle.
[Animation]

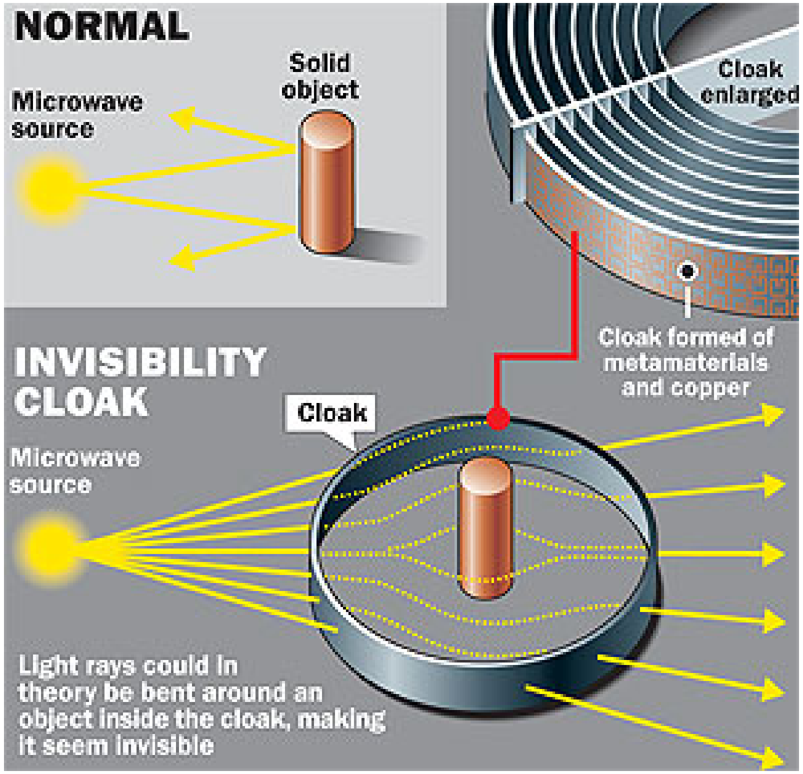
Total Internal Reflectation

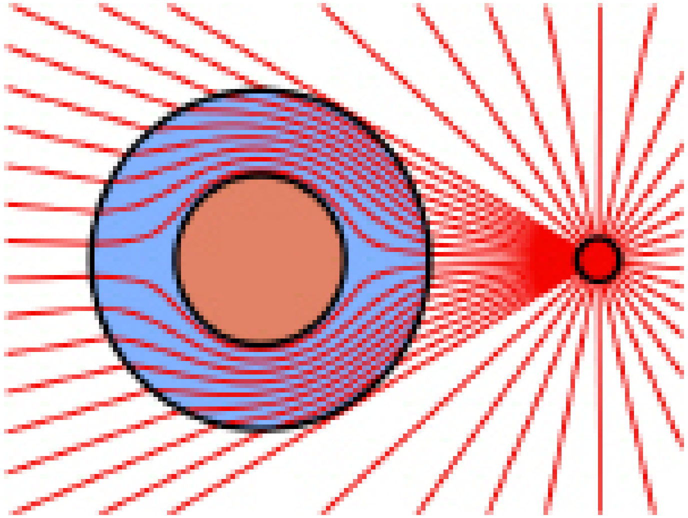


$$\theta_{\rm crit} = \arcsin\left(\frac{n_2}{n_1}\sin\theta_2\right) = \arcsin\frac{n_2}{n_1} = 48.6°.$$

Animation: [The world as seen by a fish.]

Invisible Cloaks?

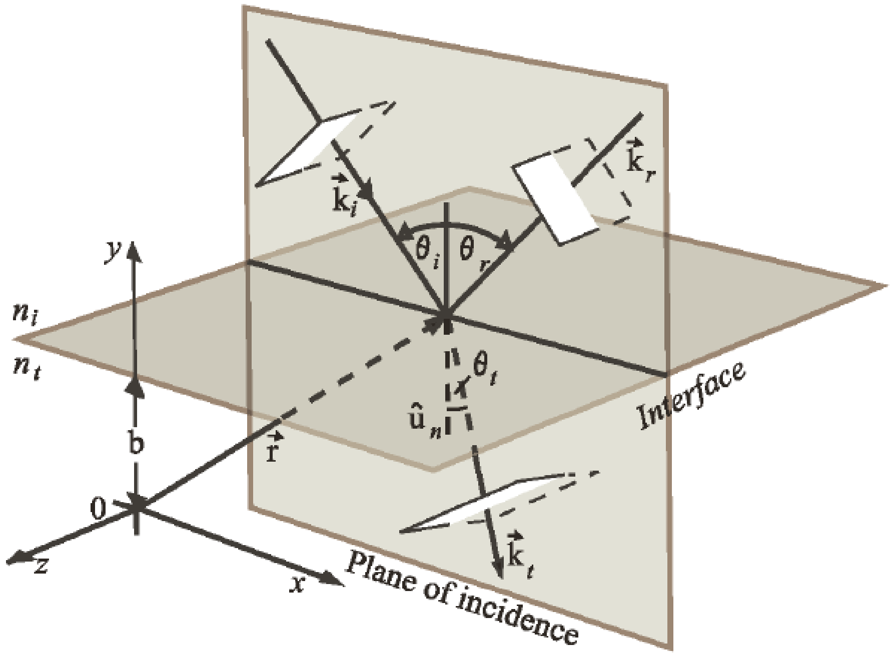


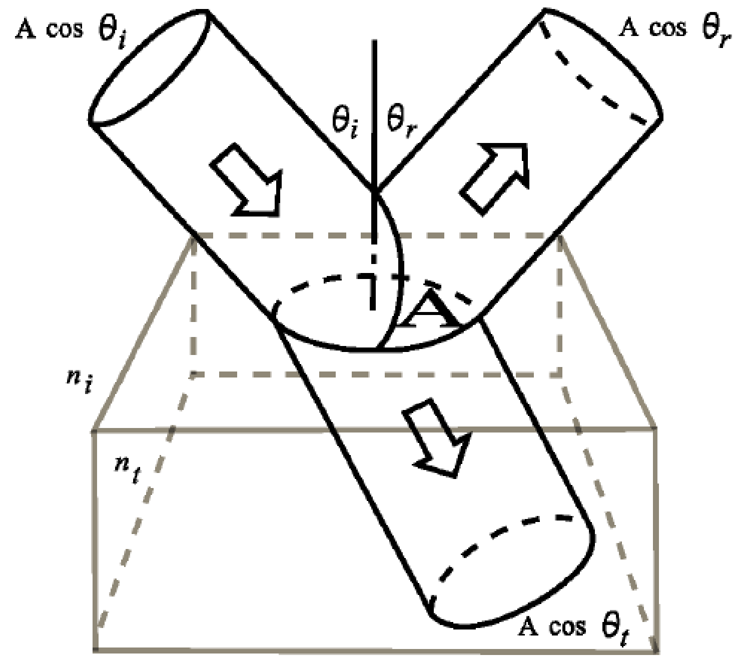


Negative Refraction Resources

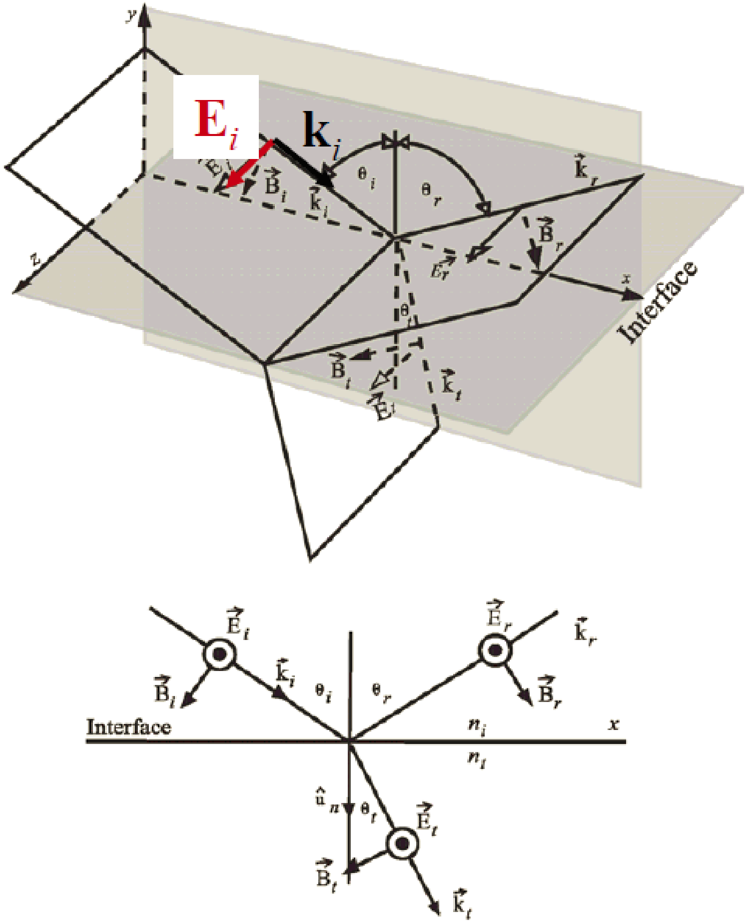
- http://en.wikipedia.org/wiki/Negative_refraction
- http://en.wikipedia.org/wiki/Metamaterial
- "Reversing Light: Negative Refraction", John Pendry and David Smith, Physics Today (Dec 2003).
- John Pendry's presentation slides
  - http://www.cleoconference.org/materials/07pendry.pdf

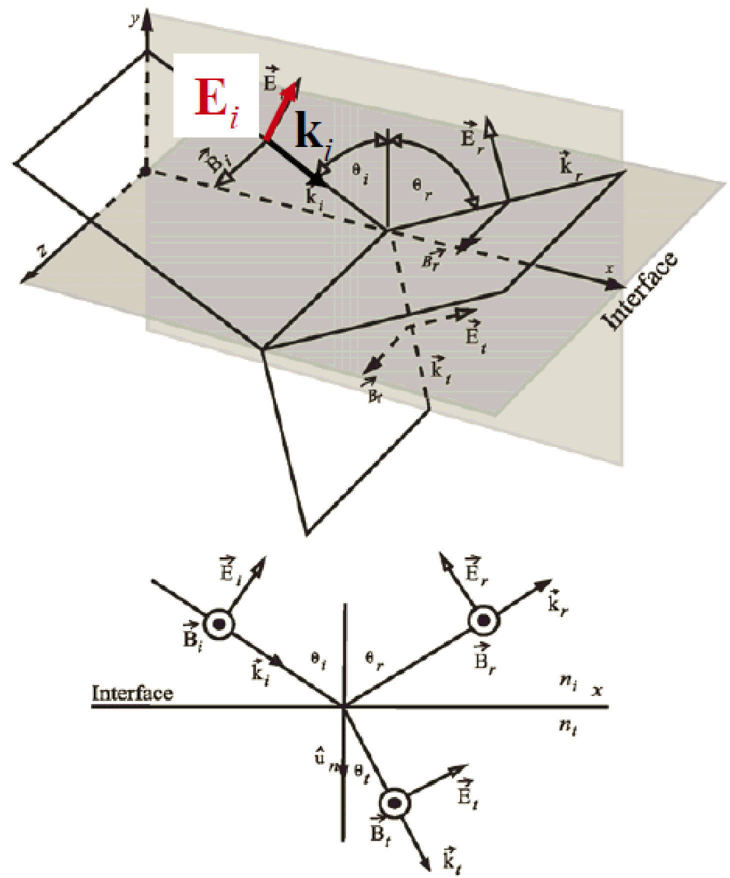
Reflection and Transmission @ dielectric interface





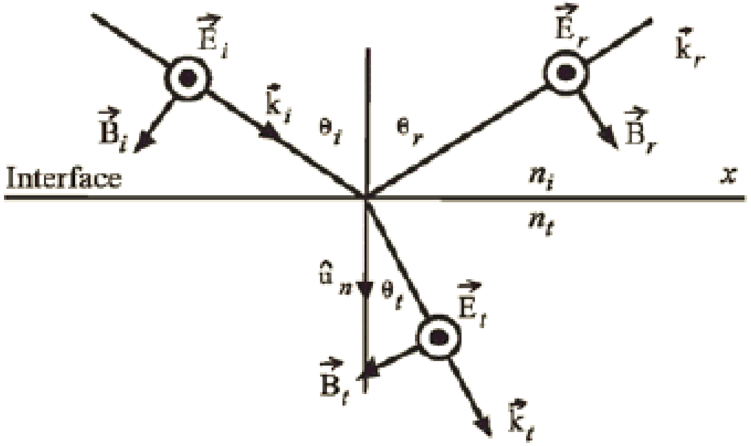
Beyond Snell's Law: Polarization?





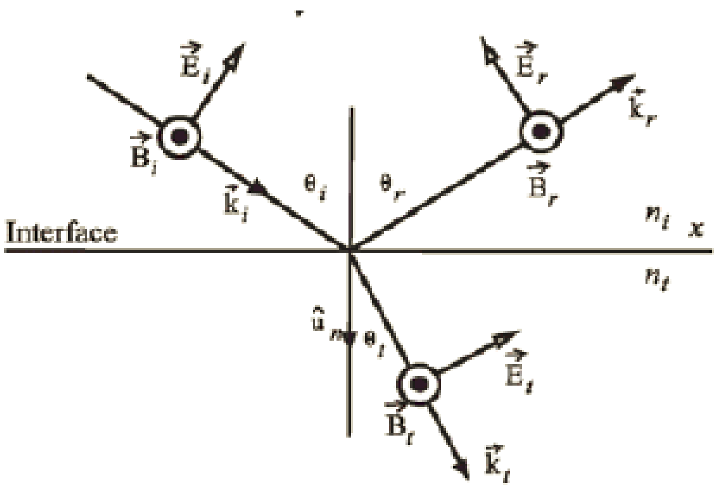
Reflection and Transmission (Fresnel's equations)

Can be deduced from the application of boundary conditions of EM waves.



$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$



$$r_\parallel = \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

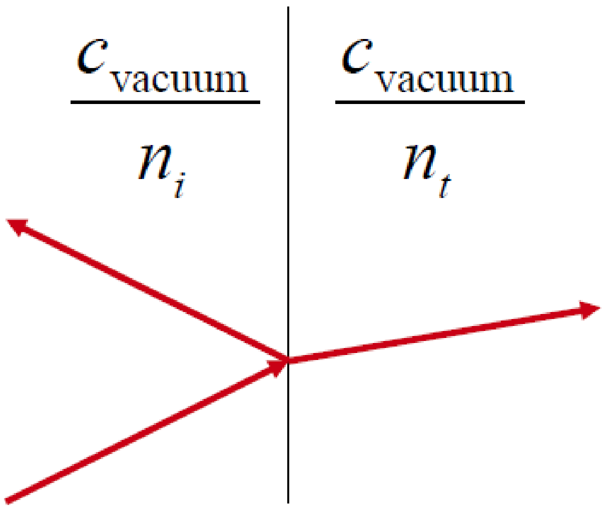
$$t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

Reflection and Transmission of Energy @ dielectric interfaces

Recall Poynting vector definition:

$$\|\mathbf{S}\| = c\varepsilon_0 \|\mathbf{E}\|^2$$

different on the two sides of the interface

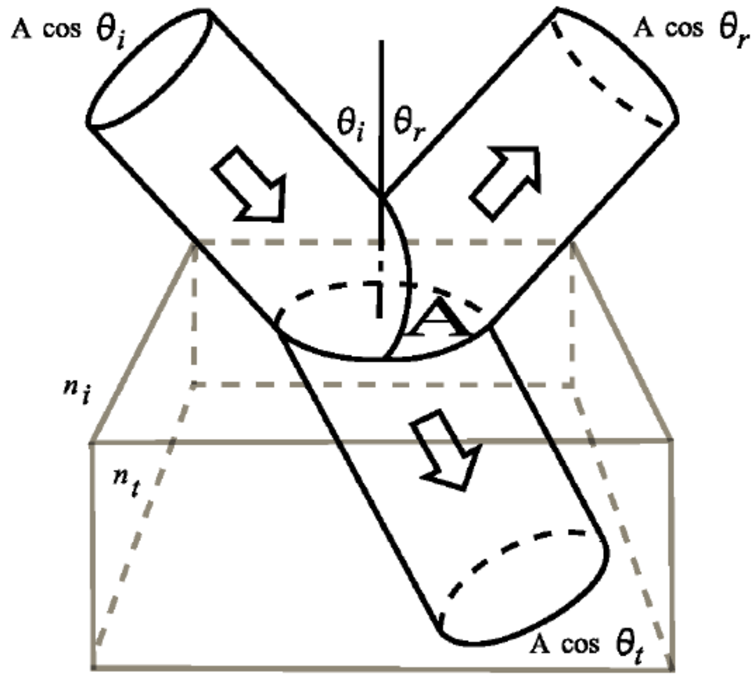


$$R = \left(\frac{E_{0r}}{E_{0i}}\right)^2 = r^2$$

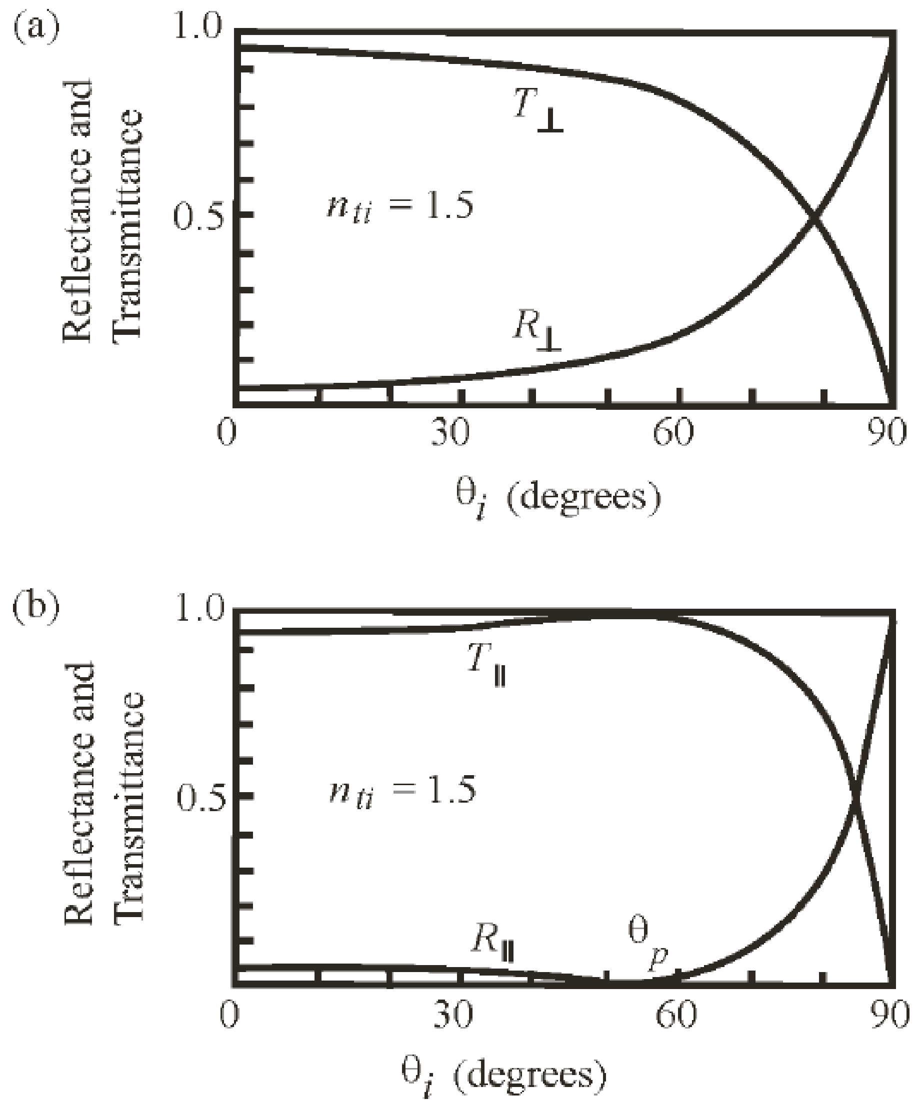
$$T = \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\left(\frac{E_{0t}}{E_{0i}}\right)^2 = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} t^2$$

Energy Conservation

$$R + T = 1, \text{ i.e. } \quad r^2 + \frac{n_t \cos \theta_t}{n_i \cos \theta_i} t^2 = 1$$



Reflectance and Transmittance @ dielectric interfaces



Normal Incidence

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$r_\parallel = \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

$$t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

Note: independent of polarization

$\theta_i = 0$ and $\theta_t = 0$ →

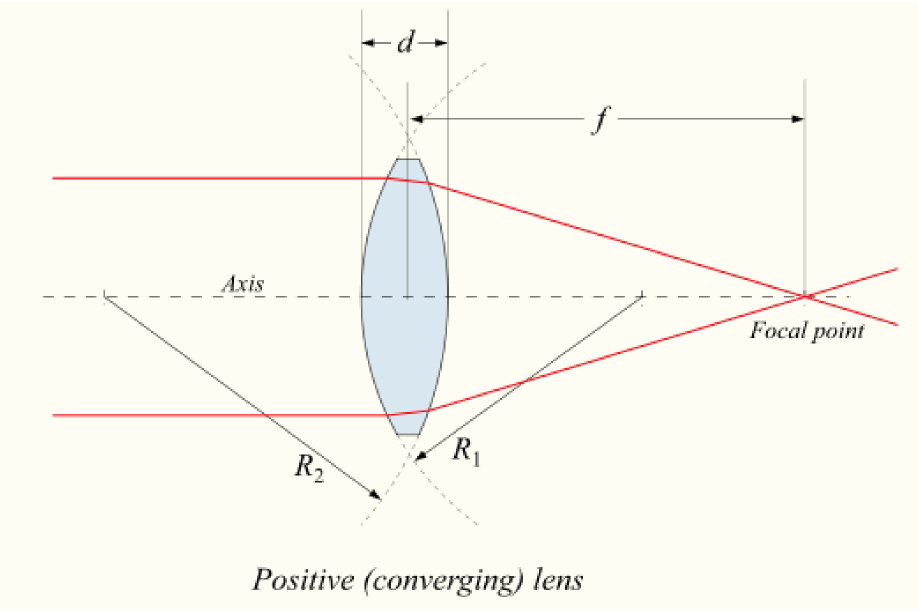
$$r_\perp = r_\parallel = \frac{n_t - n_i}{n_t + n_i}$$

$$t_\perp = t_\parallel = \frac{2n_i}{n_t + n_i}$$

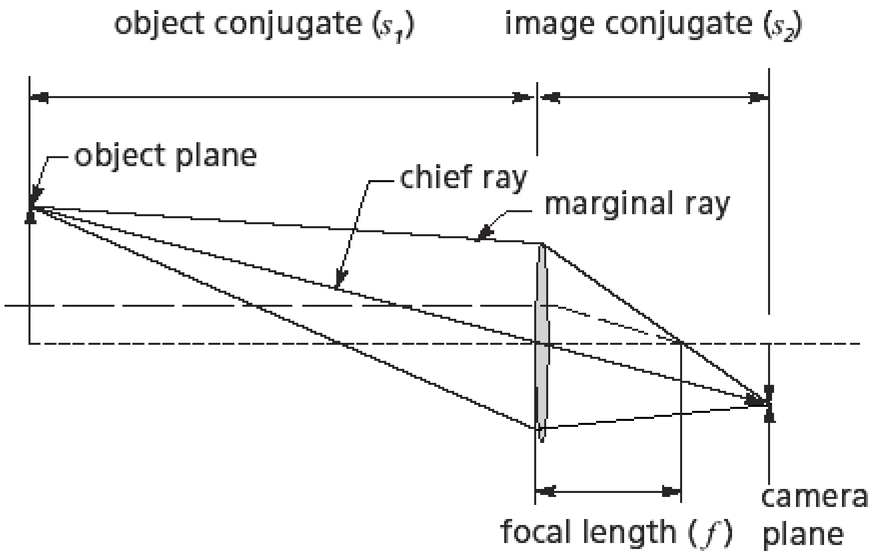
$$R_\perp = R_\parallel = \left(\frac{n_t - n_i}{n_t + n_i}\right)^2$$

$$T_\perp = T_\parallel = \frac{4n_t n_i}{(n_t + n_i)^2}$$

Thin Lens



Positive (converging) lens



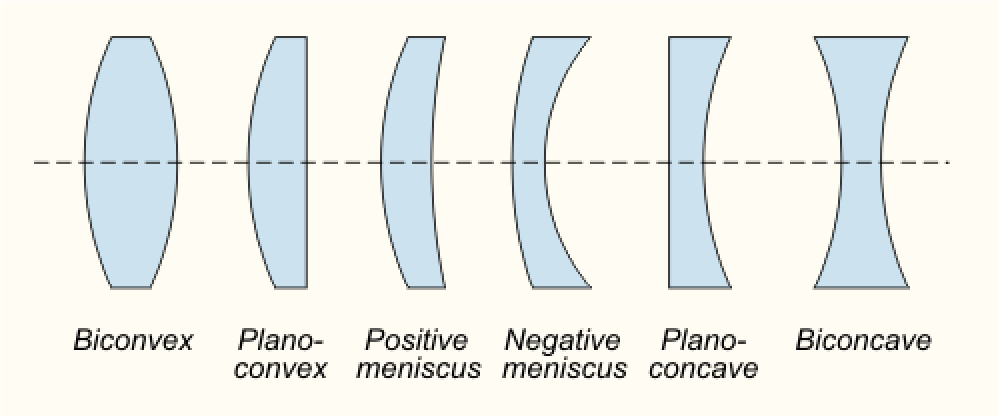
$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right],$$

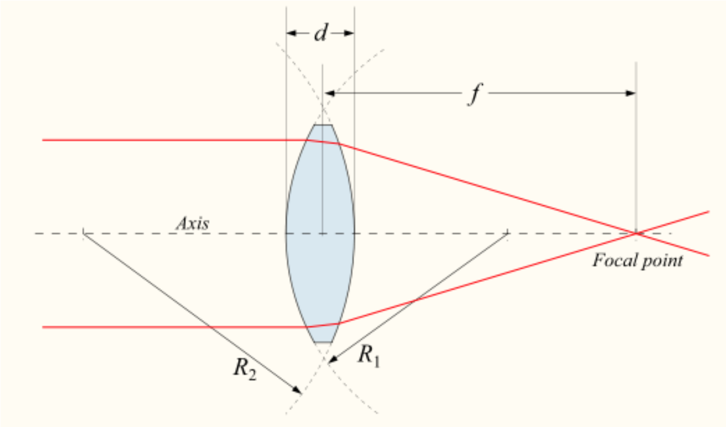
$$\frac{1}{f} \approx (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right].$$

$$\frac{1}{f} = \frac{1}{S_1} + \frac{1}{S_2}$$

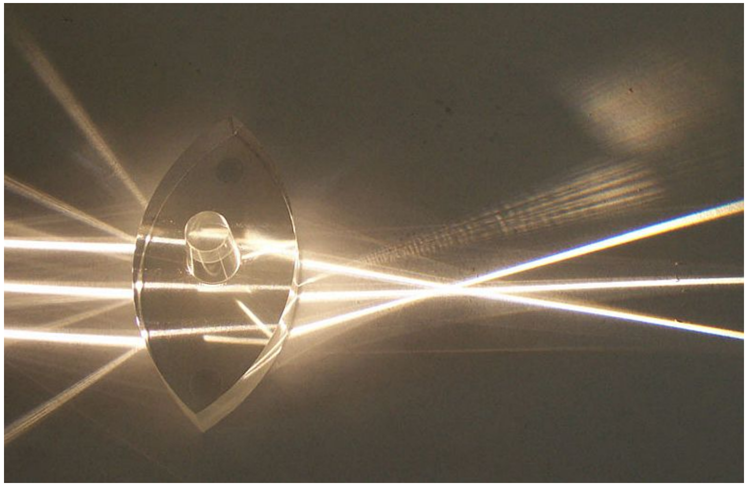
$$M_T = \frac{S_2}{S_1}$$

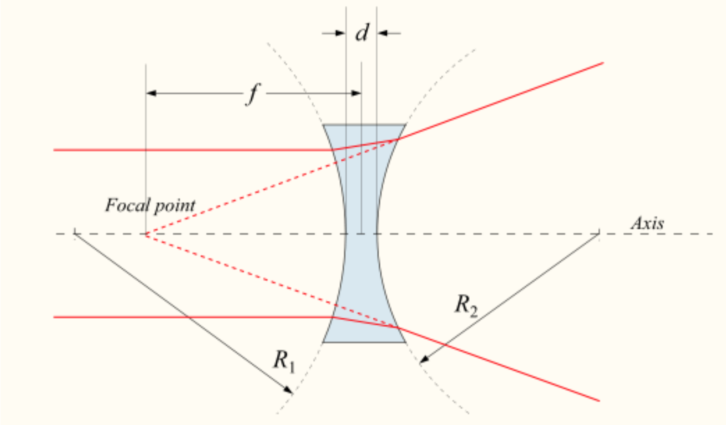
Simulation
Thin Lens: link
Thin lens combination:
http://silver.neep.wisc.edu/~shock/tools/ray.html





Positive (converging) lens





Negative (diverging) lens