

Physics 471 Summary for Second Midterm Exam
11/9/2009

Quantum States

1. The physical state of a quantum-mechanical system is represented by a vector in a Hilbert space, sometime called the “state vector.” In Dirac notation, we label the state vector by the “ket” $|\Psi\rangle$.

2. We can represent the state vector by its components using any complete set of orthonormal basis states, the same way we do with ordinary vectors. If the basis set is discrete, we write $|\Psi\rangle = \sum_{n=1}^{\infty} c_n |n\rangle$, where the vector “components” c_n can be found from taking the inner product: $c_n = \langle n | \Psi \rangle$. If the basis is continuous, the sum is replaced by an integral: $|\Psi\rangle = \int_{-\infty}^{\infty} c(q) |q\rangle$, where $c(q) = \langle q | \Psi \rangle$. Our familiar “wave function” is just the representation of the state vector in the position basis: $\Psi(x, t) = \langle x | \Psi(t) \rangle$

3. In terms of wave functions, the inner product of two quantum states is defined by

$$\langle \Psi | \Phi \rangle = \int_{-\infty}^{\infty} \Psi^*(x, t) \Phi(x, t) dx$$

4. An example of a continuous basis is the set of momentum eigenstates, which I’ll label by: $\hat{p} |k\rangle = \hbar k |k\rangle$. In that basis we can write $|\Psi\rangle = \int_{-\infty}^{\infty} \Phi(k) |k\rangle$ where $\Phi(k) = \langle k | \Psi \rangle$. If we use wavefunctions, we get our old Fourier transform expressions:

$$\Psi(x, t = 0) = \int_{-\infty}^{\infty} \Phi(k) \frac{1}{\sqrt{2\pi}} \exp(ikx) dk \quad \text{where} \quad \Phi(k) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-ikx) \Psi(x, t = 0) dx.$$

Physical Observables

1. Physical observables are represented by Hermitian operators, which have a number of important properties:

- i) their eigenvalues are real
- ii) they have a complete set of eigenstates
- iii) eigenstates with different eigenvalues are orthogonal
- iv) eigenstates with the same eigenvalue can be made orthogonal by using the Gram-Schmidt procedure

There is one catch in infinite-dimensional vector spaces. When the spectrum is continuous, the eigenstates are not normalizable – i.e. they are not in our Hilbert space. Fortunately that doesn’t matter, because we can still use them as basis vectors in the ways described above.

2. If we measure any physical observable, the measurement yields one of the eigenvalues of the corresponding operator. If the eigenvalue spectrum is discrete, the probability to obtain the n^{th} eigenvalue as a result of the measurement is $P_n = |c_n|^2 = |\langle n | \Psi \rangle|^2$. If the spectrum is continuous, the probability to obtain a result in the interval $[q_a, q_b]$ is

$$P[q_a, q_b] = \int_{q_a}^{q_b} |c(q)|^2 dq = \int_{q_a}^{q_b} |\langle q | \Psi \rangle|^2 dq$$

3. When we complete a measurement represented by the Hermitian operator \hat{A} and obtain the n^{th} eigenvalue of \hat{A} , then the quantum system has “collapsed” into the corresponding eigenstate of \hat{A} . Repeated measurements of \hat{A} will produce the same result. If we then measure a second observable represented by another Hermitian operator \hat{B} with $[\hat{A}, \hat{B}] \neq 0$, the measurement of \hat{B} will disturb the system and will change the result of a further measurement of \hat{A} . In this case \hat{A} and \hat{B} are said to represent “incompatible observables.” If, on the other hand, $[\hat{A}, \hat{B}] = 0$, then the measurement of \hat{B} does not change further measurements of \hat{A} . In fact, we can find simultaneous eigenstates of \hat{A} and \hat{B} in that case.

4. If our Hilbert space has a finite number of dimensions N , it is convenient to choose a basis and represent quantum states by column vectors whose entries are the vector components in that basis. Operators are represented by matrices, with matrix elements defined by $\hat{A}_{mn} = \langle m | \hat{A} | n \rangle$. You know how to find the eigenvectors and eigenvalues of \hat{A} ; let's label them this way: $\hat{A} | a_n \rangle = \alpha_n | a_n \rangle$. If your system is in the state $|\Psi\rangle$, then a measurement of \hat{A} will then produce the result α_n with probability $|c_n|^2 = |\langle a_n | \Psi \rangle|^2$. The average (expectation) value of \hat{A} can be calculated two ways (the first is easier to calculate, but the second contains more information): $\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle = \sum_{n=1}^N \alpha_n |c_n|^2$.

5. The Hamiltonian is a special operator. It not only represents the energy of the system, but it dictates the time dependence of the quantum states. If the energy spectrum is discrete, we first write the initial quantum state as: $|\Psi(t=0)\rangle = \sum_{n=1}^{\infty} c_n |n\rangle$. Then the state

for all future times is $|\Psi(t)\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \exp\left(\frac{-iE_n t}{\hbar}\right)$.

If you aren't yet comfortable with Dirac notation, you can write this equation in terms of wavefunctions:

$$\Psi(x, t) = \sum_{n=0}^{\infty} c_n \psi_n(x) \exp\left(\frac{-iE_n t}{\hbar}\right), \text{ where the probability to measure energy } E_n \text{ is } |c_n|^2.$$