Scalar Diffraction Theory and Basic Fourier Optics

[Hecht 10.2.4-10.2.6, 10.2.8, 11.2-11.3 or Fowles Ch. 5]

Scalar Electromagnetic theory:

$$\boxed{u(P, t) = Re[U(P)e^{-j\omega t}]} \quad \text{monochromatic wave}$$

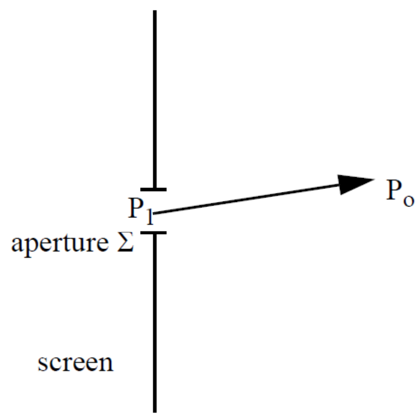
P : position $\quad t$: time $\quad \omega = 2\pi\nu$: optical frequency

$u(P, t)$ represents the E or H field strength for a particular transverse polarization component

$U(P)$: represents the complex field amplitude

$U(P) = u(P)e^{-j\phi(P)} \qquad u(P)$: real

Diffraction:



Approximations:

1. We impose the boundary condition on U, that $U = 0$ on the screen.
2. The field in the aperture Σ is not affected by the presence of the screen.

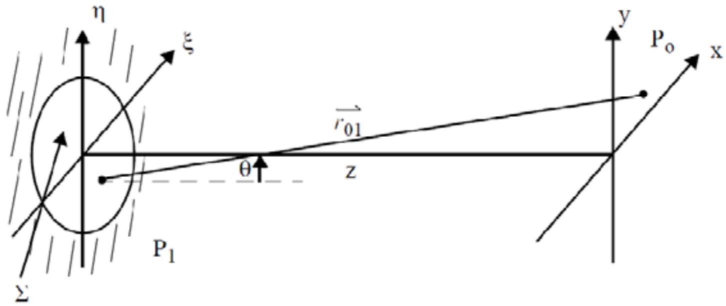
$$U(P_o) = \frac{1}{j\lambda}\iint_{\Sigma} U(P_1) \underbrace{\frac{\exp(jkr_{01})}{r_{01}}}_{\text{expanding spherical}} ds$$

$[r_{01} \gg \lambda]$

This equation expresses the Huygens-Fresnel principle: The observed field is expressed as a superposition of point sources in the aperture, with a weighting factor $\boxed{\frac{U(P_1)}{j\lambda}}$.

Fresnel approximation

Huygens-Fresnel integral in rectangular coordinates:



$$r_{01} = [z^2 + (x-\xi)^2 + (y-\eta)^2]^{1/2}$$

The Fresnel approximation involves setting: $r_{01} \cong z$ in the denominator, and

$$r_{01} \simeq z\left[1 + \frac{1}{2}\frac{(x-\xi)^2}{z} + \frac{1}{2}\frac{(y-\eta)^2}{z}\right] \quad \text{in exponent}$$

This is equivalent to the paraxial approximation in ray optics.

$$U(x,y) = \frac{\exp(jkz)}{j\lambda z}\int\int_{-\infty}^{\infty} d\xi d\eta U(\xi,\eta)\exp\left\{\frac{jk}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\} \qquad \text{(A)}$$

Let's examine the validity of the Fresnel approximation in the Fresnel integral. The next higher order term in exponent must be small compared to 1. So the valid range of the Fresnel approximation is:

$$z^3 \gg \frac{\pi}{4\lambda}[(x-\xi)^2 + (y-\eta)^2]^2_{max}$$

For field sizes of 1 cm, $\lambda = 0.5\mu m$, we find $z \gg 25$ cm.

Actually we should look at the effect on the total integral. Upon closer analysis, it is found that the Fresnel approximation holds for a much closer z. This is referred to as the "near-field region".

Farther out in z, we can approximate the quadratic phase as flat

$$z \gg \frac{k(\xi^2 + \eta^2)_{max}}{2}$$

This region is referred to as the "far-field" or Fraunhofer region.

$$U(x,y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} \underbrace{\int\int d\xi d\eta U(\xi,\eta)\exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta)\right]}_{\mathcal{F}\{U(\xi,\eta)\}\big|_{f_x = \frac{x}{\lambda z}, f_y = \frac{y}{\lambda z}}}$$

Now this is exactly the Fourier transform of the aperture distribution with

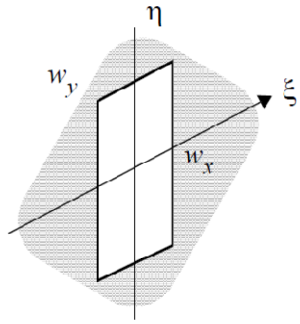
$$f_x = \frac{x}{\lambda z} \qquad f_y = \frac{y}{\lambda z}$$

The Fraunhofer region is farther out. For the field size of 1 cm, and $\lambda = 0.5\mu m$, we find the valid range of $z \gg 150$ meters!

Again, examining the full integral, Fraunhofer is actually accurate and usable to much closer distances.

Examples

A rectangular aperture, illuminated by a normally incident plane wave:



$$t_A = rect\left(\frac{\xi}{2w_x}\right)rect\left(\frac{\eta}{2w_y}\right)$$

With plane wave illumination, we have: $U(\xi, \eta) = t_A(\xi, \eta)$

$$\therefore \qquad U(x, y, z) = \frac{e^{jkz}e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z}\mathcal{F}[U]\Bigg|_{\substack{f_x = \frac{x}{\lambda z} \\ f_y = \frac{y}{\lambda z}}}$$

$$= \frac{e^{jk\left[z + \frac{x^2+y^2}{z}\right]}}{j\lambda z} A\text{sinc}\left(\frac{2w_x x}{\lambda z}\right) \text{sinc}\left(\frac{2w_y y}{\lambda z}\right)$$

$$A \equiv 4w_x w_y$$

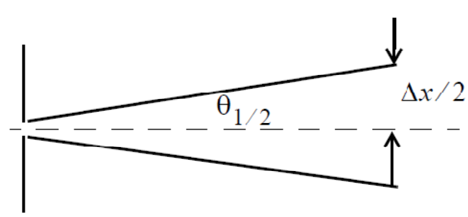
Recall $\boxed{\text{sinc}(x) \equiv \frac{\sin \pi x}{\pi x}}$. The observable is intensity $I = |U|^2$.

$$I = \frac{A^2}{\lambda^2 z^2}\text{sinc}^2\left(\frac{2w_x x}{\lambda z}\right)\text{sinc}^2\left(\frac{2w_y y}{\lambda z}\right)$$

The width of the central lobe of the diffraction pattern is

$$\boxed{\Delta x = \frac{\lambda z}{w_x}}$$

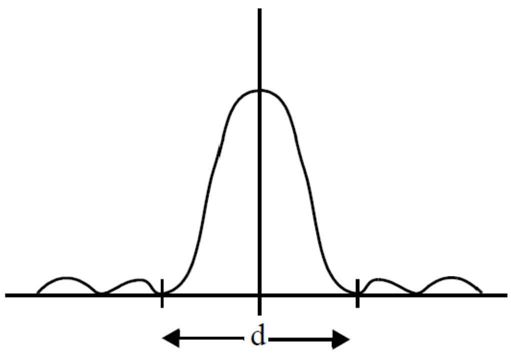
The diffraction half angle $\theta_{1/2} \cong \frac{\Delta x}{2} = \frac{\lambda}{2w_x}$

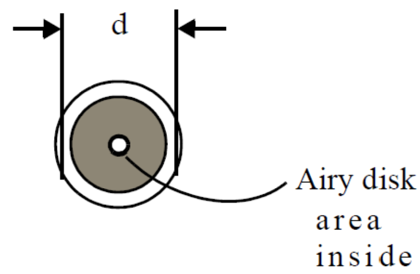


For a circular aperture with radius w : $t_A = circ\left(\frac{q}{w}\right)$ $\quad q^2 \equiv \xi^2 + \eta^2$ radial coordinates

In circular coordinates, we use the Fourier - Bessel transform: $\mathcal{B}\{U(q)\}$ gives immediately:

$$I(r) = \left(\frac{A}{\lambda z}\right)^2 \left[2\frac{J_1(kwr/z)}{kw(r/z)}\right]^2 \quad \text{“Airy pattern”}$$





$$\boxed{d = 1.22\frac{\lambda z}{w}} \quad \text{diameter of Airy disk}$$

Note (see also Fowles Ch. 5):

To calculate the diffraction pattern of a circular aperture, we can choose y as the variable of integration. If R (**w** in the above figure) is the radius of the aperture, then the element of area is taken to be a strip of width dy and length $2\sqrt{R^2 - y^2}$.

The amplitude distribution of the diffraction pattern is then given by

$$U = Ce^{ikr_0} \int_{-R}^{R} e^{iky\sin(\theta)} 2\sqrt{R^2 - y^2}\, dy\,.$$

We introduce the quantities u and ρ defined by $u = y/R$ and $\rho = kR\sin(\theta)$. The integral then becomes

$$\int_{-1}^{+1} e^{i\rho u}\sqrt{1-u^2}\, du\,.$$

This is a standard integral. Its value is $\pi J_1(\rho)/\rho$ where J_1 is the Bessel function of the first kind, order one. The ratio $J_1(\rho)/\rho \to \frac{1}{2}$ as $\rho \to 0$. The irradiance/intensity distribution is therefore given by

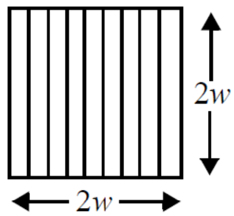
$$I = |U|^2 = I_0\left[\frac{2J_1(\rho)}{\rho}\right]^2.$$

The diffraction pattern is circularly symmetric and consists of a bright central disk surrounded by concentric circular bands of rapidly diminishing intensity. The bright central area is know as the Airy disk. It extends to the first dark ring whose size is given by the first zero of the Bessel function, namely, $\rho = 3.832$. The angular radius of the first dark ring is thus given by

$$\sin\theta = \frac{3.832}{kR} = \frac{1.22\lambda}{D} \approx \theta$$

which is valid for small values of θ (in radians). Here D=2R is the diameter of the aperture.

Diffraction grating (transmission)

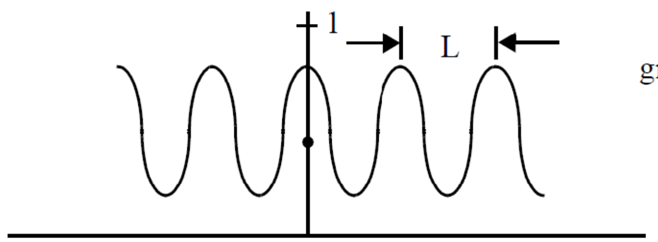


sinusoidal amplitude

$$t_A = \left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right]\mathrm{rect}\left(\frac{\xi}{2w}\right)\mathrm{rect}\left(\frac{\eta}{2w}\right)$$

m: peak to peak amplitude change $0 \le m \le 1$

f_0: grating spatial frequency

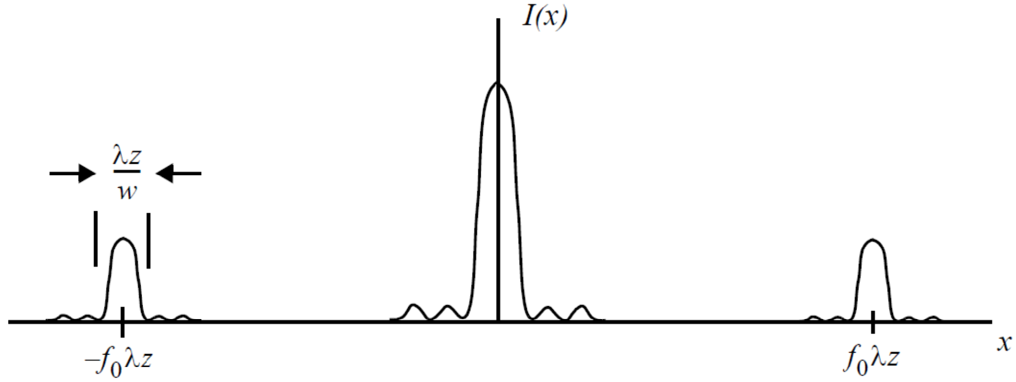


grating period

$$L = \frac{1}{f_0}$$

By convolution, the diffracted amplitude is

$$F\left[\frac{1}{2} + \frac{m}{2}\cos(2\pi f_0 \xi)\right] \otimes F\left[rect\frac{\xi}{2w}rect\frac{\eta}{2w}\right]$$



$$I(x,y) \cong \left(\frac{A}{2\lambda z}\right)^2 \mathrm{sinc}^2\frac{2wy}{\lambda z}\left\{\mathrm{sinc}^2\left(\frac{2wx}{\lambda z}\right) + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x + f_0\lambda z)\right] + \frac{m^2}{4}\mathrm{sinc}^2\left[\frac{2w}{\lambda z}(x - f_0\lambda z)\right]\right\}$$

We have neglected interference terms between orders.

Compared to the square aperture, which has the central peak with intensity I_o, we now have:

$$\frac{1}{4}I_0 \;:\; \text{zero order}$$

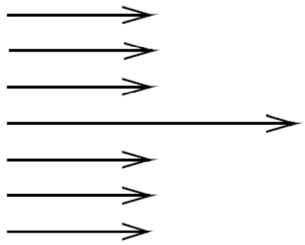
$$\frac{m^2}{16}I_0 \;:\; \pm 1 \text{ order}$$

The "resolving power" of the grating $R \equiv \dfrac{\text{peak separation}}{\text{peak width}}$

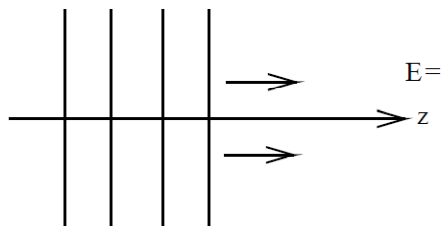
$$R = \frac{f_0 \lambda z}{\lambda z / w} = f_0 w = \frac{w}{L} = [\#\text{ grating periods}]$$

Wave Optics of Lenses

Set of rays parallel to axis



Plane Wave

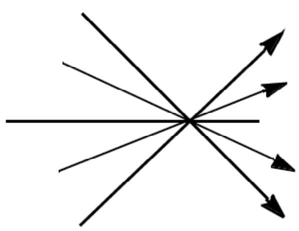


$$E = E_\circ \cos(kz - \omega t)$$

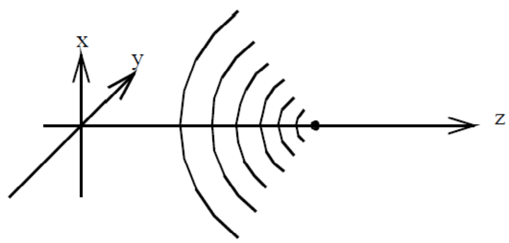
$$k = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f$$

Rays converging to a focus



converging spherical wave



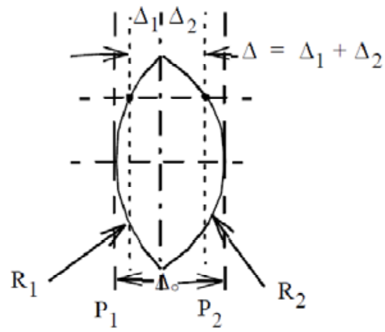
At a given z-plane, the spherical wave has constant phase around circles. The form of the spherical wave is $\cos\left[-\frac{k(x^2+y^2)}{2z_\circ}\right]$ for a spherical wave converting to the point $z_\circ$ on the axis. A lens modifies the wave front, for example from planar to spherical.

How does this happen?

Optical Path Difference

Optical waves travel more slowly in the glass since $n > 1$. In glass, the wave is delayed by an amount as if it travelled a distance nl in free space. If $l = l(x,y)$ [or $n = n(x,y)$] then the delay varies with (x,y) so the wavefront gets distorted.

We can analyze the lens in terms of its phase-delay. The light propagates in the glass as $\cos(knz) = \cos\phi$, where $\phi = knz$ is the phase delay.

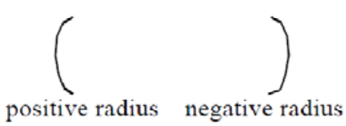


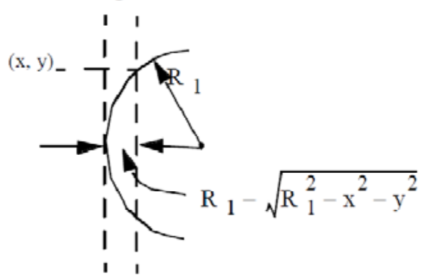
In propagating from plane P_1 to P_2, the light travels a distance $\Delta = \Delta_1 + \Delta_2$ in the glass and a distance $\Delta_\circ - \Delta$ in air, where $\Delta_\circ$ is the thickness at the thickest part of the lens. The phase delay depends on (x, y):

$$\phi(x, y) = kn\Delta(x,y) + k[\Delta_\circ - \Delta(x, y)]$$

$$= k\Delta_\circ + k(n-1)\Delta(x,y)$$

We can calculate Δ, assuming spherical surfaces. Recall the sign convention for the surface radii:





From this diagram, we can readily obtain

$$\Delta(x, y) = \Delta_\circ - \left[R_1 - \sqrt{R_1^2 - x^2 - y^2}\right] + \left[R_2 - \sqrt{R_2^2 - x^2 - y^2}\right]$$

$$= \Delta_\circ - R_1\left[1 - \sqrt{1 - \left(\frac{x^2 + y^2}{R_1^2}\right)}\right] + R_2\left[1 - \sqrt{1 - \left(\frac{x^2 + y^2}{R_2^2}\right)}\right]$$

In the <u>paraxial approximation</u> $(x^2 + y^2) \ll R_{1,2}^2$, so

$$\sqrt{1 - \left(\frac{x^2 + y^2}{R_{1,2}^2}\right)} \cong 1 - \left(\frac{x^2 + y^2}{2R_{1,2}^2}\right) \qquad \text{, thus}$$

$$\Delta \cong \Delta_\circ - \left(\frac{x^2 + y^2}{2}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

This gives a phase delay:

$$\phi(x, y) = k\Delta_\circ + k(n-1)\left[\Delta_\circ - \left(\frac{x^2 + y^2}{2}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]$$

Apart from the constant delay $kn\Delta_\circ$, the phase delay is:

$$\boxed{\phi(x, y) = -k(n-1)\left(\frac{x^2 + y^2}{2}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

A plane wave incident on the lens has a constant phase. After passing through the lens, the phase is given above. This has the form of a spherical wave, converging to a point at a distance f, where

$$\boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)},$$

f is the focal length of the lens. This expression is identical to what we found from the ray optics analysis.

Diffraction Theory of a Lens

We have previously seen that light passing through a lens experiences a phase delay given by:

$$\varphi(x,y) = \exp\left[-jk(n-1)\left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right] \quad \text{(neglecting the constant phase)}$$

The focal length, f is given by:

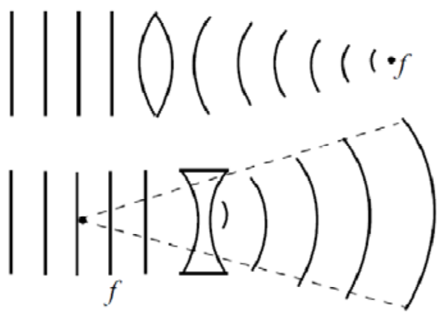
$$\boxed{\frac{1}{f} = (n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} \quad \text{The "lens makers formula"}$$

The transmission function is now:

$$\boxed{\varphi(x,y) = \exp\left[-j\frac{k}{2f}(x^2+y^2)\right]}$$

This is the paraxial approximation to the spherical phase

Note: the incident plane-wave is converted to a spherical wave converging to a point at f behind the lens (f positive) or diverging from the point at f in front of lens (f negative).



Diffraction from the lens pupil

Suppose the lens is illuminated by a plane wave, amplitude A. The lens "pupil function" is $P(x,y)$.

The full effect of the lens is $U_l'(x,y) = \varphi(x,y)P(x,y)$

$$U_l'(x,y) = P(x,y)\exp\left[-j\frac{k}{2f}(x^2+y^2)\right]$$

We now use the Fresnel formula to find the amplitude at the "back focal plane" $z = f$

$$U_f(u,v) = \frac{\exp\left[j\frac{k}{2f}(u^2+v^2)\right]}{j\lambda f} \times e^{jkf}\int\int_{-\infty}^{\infty} dxdy\, U_l'(x,y)\exp\left[j\frac{k}{2f}(x^2+y^2)\right]\exp\left[-j\frac{2\pi}{\lambda f}(xu+yv)\right]$$

The phase terms that are quadratic in x^2+y^2 cancel each other.

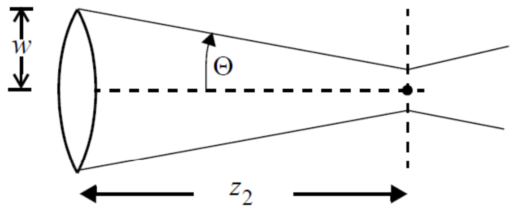
$$U_f(u,v) = \frac{\exp\left[j\frac{k}{2f}(u^2+v^2)\right]}{j\lambda f}e^{jkf}\int\int_{-\infty}^{\infty} dxdy\, P(x,y)\exp\left[-j\frac{2\pi}{\lambda f}(xu+yv)\right] \quad \text{(B)}$$

This is precisely the Fraunhofer diffraction pattern of P ! Note that a large z criterion *does not* apply here.

The focal plane amplitude distribution is a Fourier transform of the lens pupil function *P(x,y)*, multiplied by a quadratic phase term. However, the intensity distribution is exactly

$$I_f(u,v) = \frac{A^2}{\lambda^2 f^2}\left|\mathcal{F}[P(x,y)]\right|^2 \qquad f_x = \frac{u}{\lambda f} \quad f_y = \frac{v}{\lambda f}$$

Example: a circular lens, with radius w



$$P = \text{circ}\left(\frac{q}{w}\right) \qquad (q^2 = x^2 + y^2)$$

$$\text{let} \quad h(r) = \mathcal{F}[P(\lambda z_2 q)] = \mathcal{F}\left[\text{circ}\left(\frac{\lambda z_2 q}{w}\right)\right] \qquad (r^2 = u^2 + v^2)$$

$$= \frac{A}{\lambda z_2}\left[2\frac{J_1(2\pi w r/\lambda z_2)}{2\pi w r/\lambda z_2}\right]$$

$$|h(r)|^2 = \frac{A^2}{\lambda^2 z_2^2}\left[2\frac{J_1(2\pi w r/\lambda z_2)}{2\pi w r/\lambda z_2}\right]^2$$

The spot diameter is $d = 1.22\frac{\lambda f}{w} = 1.22\frac{\lambda}{\theta}$

The resolution of the lens as defined by the "Rayleigh" criterion is $d/2 = 0.61\lambda/\theta$**. For a small angle θ,** $d/2 = 0.61\lambda/\sin\theta = 0.61\frac{\lambda}{NA}$**.**

Fowles Ch. 5

This is the criterion for Fraunhofer diffraction. If this condition does not obtain, the curvature of the wave front becomes important and the diffraction is of the Fresnel type. Similar considerations apply in the case of diffraction by an opaque object or obstacle. Then δ is the linear size of the object. (Note that Babinet's principle applies here.)

Examples of Fraunhofer and Fresnel diffraction by various types of apertures are treated in the sections that follow. Since the Fraunhofer case is, in general, mathematically simpler than the Fresnel case, Fraunhofer diffraction will be discussed first.

5.4 Fraunhofer Diffraction Patterns

The usual experimental arrangement for observing Fraunhofer diffraction is shown in Figure 5.7. Here the aperture is *coherently*

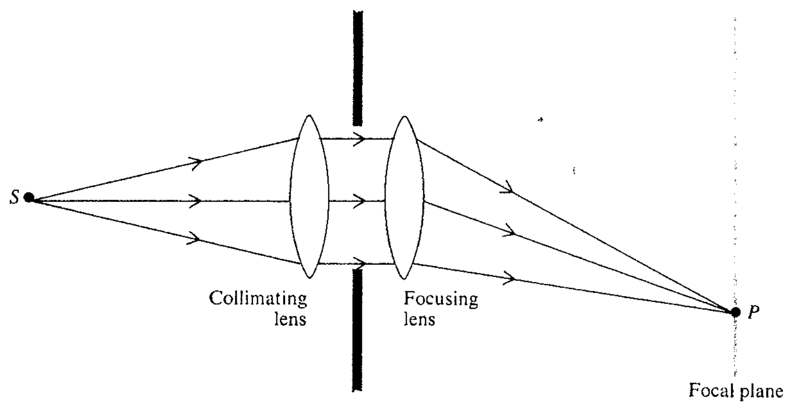


Figure 5.7. Arrangement for observing Fraunhofer diffraction.

illuminated by means of a point monochromatic source and a collimating lens. A second lens is placed behind the aperture as shown. The incident and diffracted wave fronts are therefore strictly plane, and the Fraunhofer case is rigorously valid. In applying the Fresnel-Kirchhoff formula [Equation (5.11)] to the calculation of the diffraction patterns, the following simplifying approximations are taken to be valid:

(1) The angular spread of the diffracted light is small enough for the obliquity factor [cos (**n**,**r**) − cos (**n**,**r**′)] not to vary appreciably over the aperture and to be taken outside the integral.
(2) The quantity $e^{ikr'}/r'$ is very nearly constant and can be taken outside the integral.

(3) The variation of the remaining factor e^{ikr}/r over the aperture comes principally from the exponential part, so the factor $1/r$ can be replaced by its mean value and taken outside the integral.

Consequently, the Fresnel-Kirchhoff formula reduces to the very simple equation

$$U_p = C \iint e^{ikr} \, d\mathscr{A} \tag{5.16}$$

where all constant factors have been lumped into one constant C. The formula above states that the distribution of the diffracted light is obtained simply by integrating the phase factor e^{ikr} over the aperture.

The Single Slit The case of diffraction by a single narrow slit is treated here as a one-dimensional problem. Let the slit be of length L

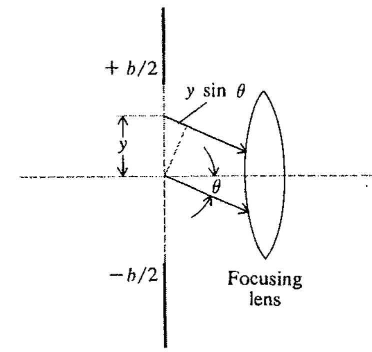


Figure 5.8. Definition of the variables for Fraunhofer diffraction by a single slit.

and of width b. The element of area is then $d\mathscr{A} = L\,dy$ as indicated in Figure 5.8. Furthermore, we can express r as

$$r = r_0 + y \sin\theta \tag{5.17}$$

where r_0 is the value of r for $y = 0$, and where θ is the angle shown. The diffraction formula (5.16) then yields

$$\begin{aligned} U &= Ce^{ikr_0} \int_{-b/2}^{+b/2} e^{iky\sin\theta} L\,dy \\ &= 2\,Ce^{ikr_0} L \frac{\sin(\frac{1}{2}kb\sin\theta)}{k\sin\theta} = C' \left(\frac{\sin\beta}{\beta}\right) \end{aligned} \tag{5.18}$$

where $\beta = \frac{1}{2}kb\sin\theta$, and $C' = e^{ikr_0} CbL$ is merely another constant.

Thus C' (sin β/β) is the total amplitude of the light diffracted in a given direction defined by β. This light is brought to a focus by the second lens, and the corresponding irradiance distribution in the focal plane is given by the expression

$$I = |U|^2 = I_0 \left(\frac{\sin \beta}{\beta}\right)^2 \tag{5.19}$$

where $I_0 = |CLb|^2$, which is the irradiance for $\theta = 0$. The distribution is plotted in Figure 5.9. The maximum value occurs at $\theta = 0$, and zero

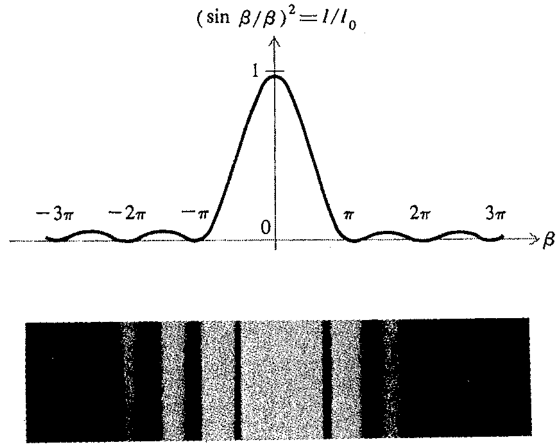


Figure 5.9. Fraunhofer diffraction pattern of a single slit.

values occur for $\beta = \pm\pi, \pm 2\pi, \ldots$, and so forth. Secondary maxima of rapidly diminishing value occur between these zero values. Thus the diffraction pattern at the focal plane consists of a central bright band. On either side there are alternating bright and dark bands. Table 5.1 gives the relative values of I of the first three secondary maxima. The first minimum, $\beta = \pi$, corresponds to

$$\sin \theta = \frac{2\pi}{kb} = \frac{\lambda}{b} \tag{5.20}$$

Table 5.1. Relative Values of the Maxima of Diffraction Patterns of Rectangular and Circular Apertures

	Rectangular	*Circular*
Central Max	1	1
1st Max	0.0496	0.0174
2d Max	0.0168	0.0042
3rd Max	0.0083	0.0016

Thus, for a given wavelength, the angular width of the diffraction pattern varies inversely with the slit width, and the amplitude of the central maximum is proportional to the area of the slit. For very narrow slits the pattern is dim but wide. It shrinks and becomes brighter as the slit is widened.

The Rectangular Aperture The case of diffraction by a single aperture of rectangular shape is treated in the same way as the single slit, except that one must now integrate in two dimensions, say x and y as shown in Figure 5.10. It is left as a problem to show that the irradiance distribution is given by the product of two single-slit distribution functions. (See Section 5.6.) The result is

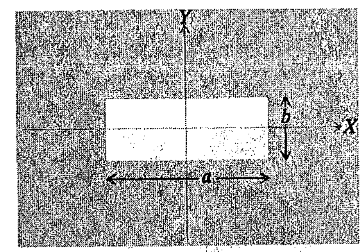


Figure 5.10. Rectangular aperture.

$$I = I_0 \left(\frac{\sin \alpha}{\alpha}\right)^2 \left(\frac{\sin \beta}{\beta}\right)^2 \tag{5.21}$$

where $\alpha = \frac{1}{2}ka \sin \phi$, $\beta = \frac{1}{2}kb \sin \theta$. The dimensions of the aperture are a and b and the angles ϕ and θ define the direction of the diffracted ray. The resulting diffraction pattern (Figure 5.11) has lines of zero irradiance defined by $\alpha = \pm\pi, \pm 2\pi, \ldots$, and $\beta = \pm\pi, \pm 2\pi \ldots$. As with the slit, the scale of the diffraction pattern bears an inverse relationship to the scale of the aperture.

The Circular Aperture To calculate the diffraction pattern of a circular aperture, we choose y as the variable of integration, as in the case of the single slit. If R is the radius of the aperture, then the element of area is taken to be a strip of width dy and length $2\sqrt{R^2 - y^2}$ (Figure 5.12).

The amplitude distribution of the diffraction pattern is then given by

$$U = Ce^{ikr_0} \int_{-R}^{+R} e^{iky \sin \theta}\, 2\sqrt{R^2 - y^2}\, dy \tag{5.22}$$

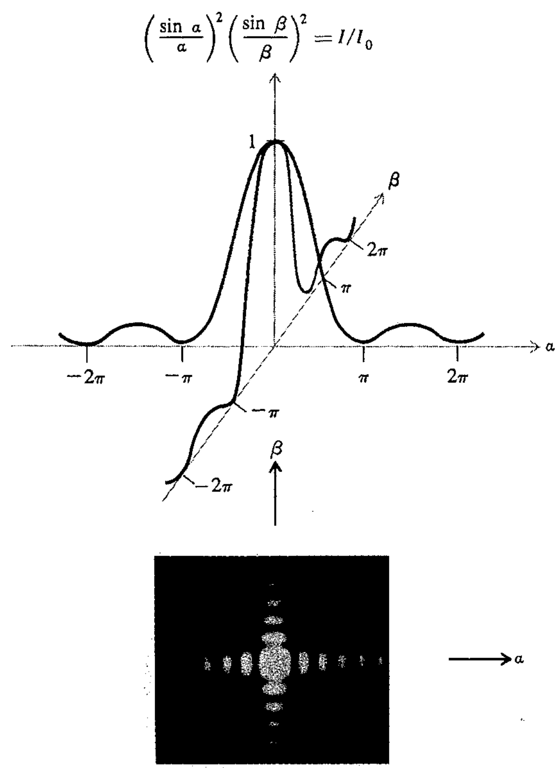


Figure 5.11. Fraunhofer diffraction pattern of a rectangular aperture.

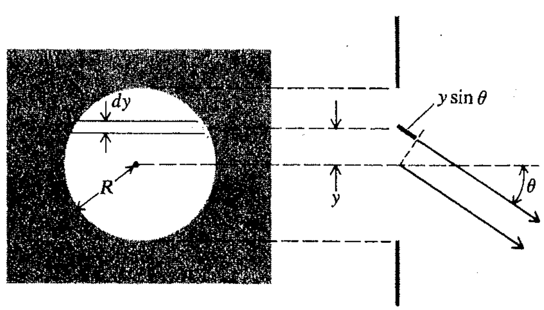


Figure 5.12. Circular aperture.

We introduce the quantities u and ρ defined by $u = y/R$ and $\rho = kR \sin\theta$. The integral in Equation 5.22 then becomes

$$\int_{-1}^{+1} e^{i\rho u} \sqrt{1-u^2}\, du \tag{5.23}$$

This is a standard integral. Its value is $\pi J_1(\rho)/\rho$ where J_1 is the Bessel function of the first kind, order one [27]. The ratio $J_1(\rho)/\rho \rightarrow \frac{1}{2}$ as $\rho \rightarrow 0$. The irradiance distribution is therefore given by

$$I = I_0 \left[\frac{2J_1(\rho)}{\rho}\right]^2 \tag{5.24}$$

where $I_0 = (C\pi R^2)^2$, which is the intensity for $\theta = 0$.

A graph of the intensity function is shown in Figure 5.13. The diffraction pattern is circularly symmetric and consists of a bright cen-

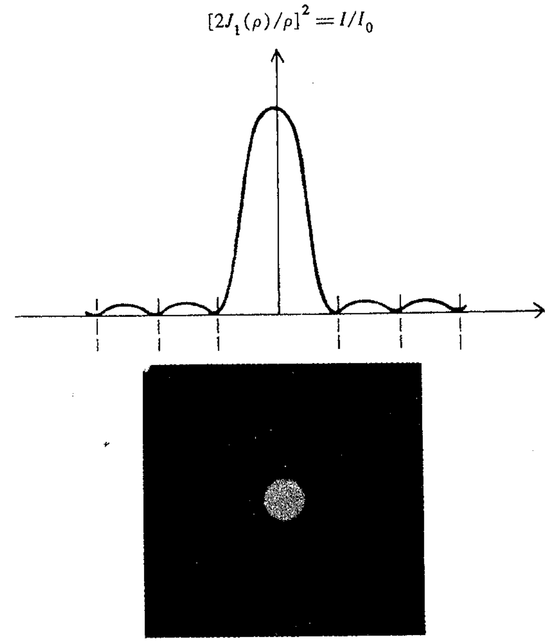


Figure 5.13. Fraunhofer diffraction pattern of a circular aperture.

tral disk surrounded by concentric circular bands of rapidly diminishing intensity. The bright central area is known as the *Airy disk.* It extends to the first dark ring whose size is given by the first zero of the Bessel function, namely, $\rho = 3.832$. The angular radius of the first dark ring is thus given by

$$\sin\theta = \frac{3.832}{kR} = \frac{1.22\lambda}{D} \approx \theta \tag{5.25}$$

which is valid for small values of θ. Here $D = 2R$ is the diameter of the aperture.

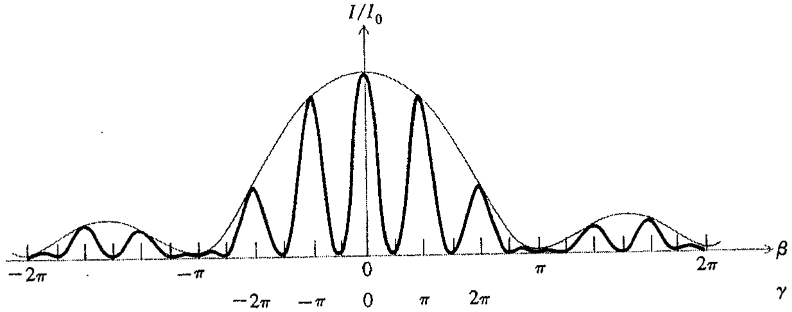


Figure 5.16. Fraunhofer diffraction pattern of a double-slit aperture.

Multiple Slits. Diffraction Gratings Let the aperture consist of a grating, that is, a large number N of identical parallel slits of width b and separation h (Figure 5.17). The evaluation of the diffractional integral is carried out in a manner similar to that of the double slit:

$$\begin{aligned}\int_{\mathscr{A}} e^{iky\sin\theta}\,dy &= \int_0^b + \int_h^{h+b} + \int_{2h}^{2h+b} + \cdots + \int_{(N-1)h}^{(N-1)h+b} e^{iky\sin\theta}\,dy \\ &= \frac{e^{ikb\sin\theta}-1}{ik\sin\theta}\left[1 + e^{ikh\sin\theta} + \cdots + e^{ik(N-1)h\sin\theta}\right] \\ &= \frac{e^{ikb\sin\theta}-1}{ik\sin\theta}\cdot\frac{1-e^{ikNh\sin\theta}}{1-e^{ikh\sin\theta}} \\ &= be^{i\beta}e^{i(N-1)\gamma}\left(\frac{\sin\beta}{\beta}\right)\left(\frac{\sin N\gamma}{\sin\gamma}\right)\end{aligned} \tag{5.29}$$

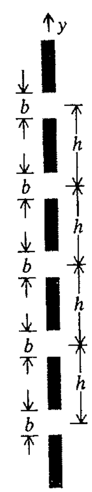


Figure 5.17. Multiple-slit aperture or diffraction grating.

where $\beta = \frac{1}{2}kb\sin\theta$ and $\gamma = \frac{1}{2}kh\sin\theta$. This yields the following intensity distribution function:

$$I = I_0\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\gamma}{N\sin\gamma}\right)^2 \tag{5.30}$$

The factor N has been inserted in order to normalize the expression. This makes $I = I_0$ when $\theta = 0$.

Again the single-slit factor $(\sin\beta/\beta)^2$ appears as the envelope of the diffraction pattern. Principal maxima occur within the envelope at $\gamma = n\pi$, $n = 0, 1, 2, \ldots$, that is,

$$n\lambda = h\sin\theta \tag{5.31}$$

which is the grating formula giving the relation between wavelength and angle of diffraction. The integer n is called the *order of diffraction.*

Secondary maxima occur near $\gamma = 3\pi/2N$, $5\pi/2N$, and so forth, and zeros occur at $\gamma = \pi/N$, $2\pi/N$, $3\pi/N$. . . . A graph is shown in Figure 5.18(a). If the slits are very narrow, then the factor $\sin\beta/\beta \approx 1$. The first few primary maxima, then, all have approximately the same value, namely, I_0.

Resolving Power of a Grating The angular width of a principal fringe, that is, the separation between the peak and the adjacent minimum, is found by setting the *change* of the quantity $N\gamma$ equal to π, that is, $\Delta\gamma = \pi/N = \frac{1}{2}kh\cos\theta\,\Delta\theta$, or

$$\Delta\theta = \frac{\gamma\lambda}{Nh\cos\theta} \tag{5.32}$$

Thus if N is made very large, then $\Delta\theta$ is very small, and the diffraction pattern consists of a series of sharp fringes corresponding to the different orders $n = 0, \pm1, \pm2$, and so forth [Figure 5.18(b), (c)]. On the other hand for *a given order* the dependence of θ on the wavelength [Equation (5.31)] gives by differentiation

$$\Delta\theta = \frac{n\,\Delta\lambda}{h\cos\theta} \tag{5.33}$$

This is the angular separation between two spectral lines differing in wavelength by $\Delta\lambda$. Combining Equation (5.32) and (5.33), we obtain the *resolving power* of a grating spectroscope according to the Rayleigh criterion, namely,

$$\mathrm{RP} = \frac{\lambda}{\Delta\lambda} = Nn \tag{5.34}$$

In words, the resolving power is equal to the number of grooves N multiplied by the order number n.