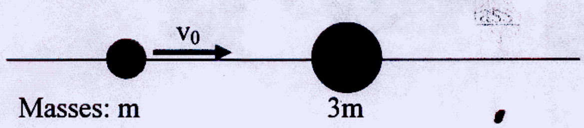
PHY 321 Quiz F Tuesday, June 30

1. Assume this is a perfectly elastic head-on collision. Calculate the final velocity of the larger mass.

Solution Key



2. Consider a rocket in interstellar space. For t < 0 the rocket is at rest. At t = 0 the rocket is ignited. During the rocket burn, the <u>mass exhaust rate</u> = μ = $-dm/dt$ is constant; and the <u>relative exhaust speed</u> is u. (Note that both μ and u are positive.) Determine the acceleration of the rocket $\boldsymbol{a}$, as a function of time. Sketch a qualitatively correct graph of $\boldsymbol{a}$ versus $\boldsymbol{t}$. Assume $m_{final} = 0.1\, m_{initial}$.

① Momentum is conserved $\quad m v_0 = m v_1 + 3m v_2$ (final velocities) 1 point

Energy is conserved $\quad \frac{1}{2} m v_0^2 = \frac{1}{2} m v_1^2 + \frac{1}{2} 3m v_2^2$ 1 point

Solve for v_2:

$v_1 = v_0 - 3v_2 \quad \longrightarrow \quad v_0^2 = (v_0 - 3v_2)^2 + 3v_2^2$

$v_0^2 = v_0^2 - 6 v_0 v_2 + 12 v_2^2$

$\underline{v_2 = \frac{1}{2} v_0}$ 1 point / The solution $v_2 = 0$ corresponds to no collision. /

3 points

②

Rocket equation $\quad dP_{total} = 0$

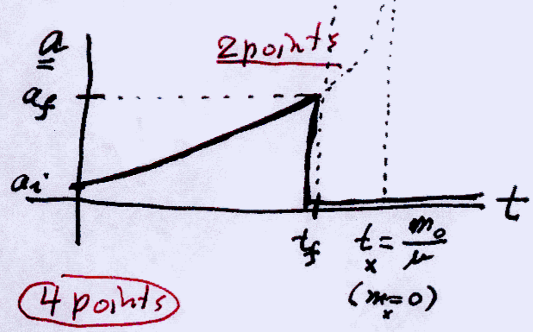
$$dP_{total} = (m - \mu dt)(v + dv) + \mu dt\,(v - u) - mv$$
$$= m\,dv - \mu u\, dt = 0$$

$$\underline{a} = \frac{dv}{dt} = \frac{\mu u}{m_0 - \mu t}$$

2 points

$$t_f = \frac{0.9\, m_0}{\mu}$$

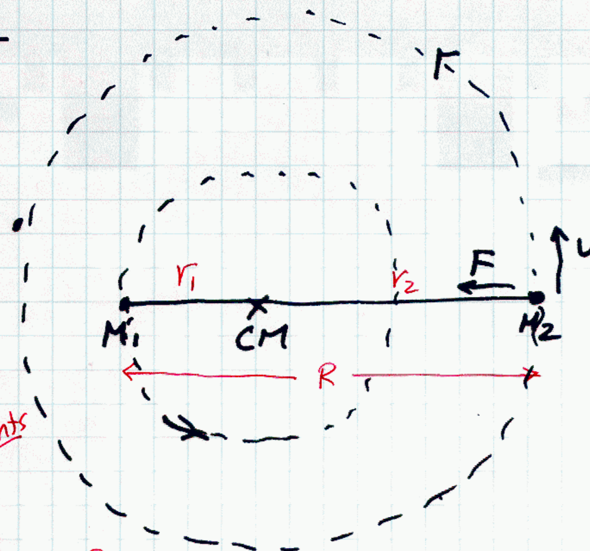
$$a_f = \frac{\mu u}{0.1\, m_0} \; ; \quad a_i = \frac{\mu u}{m_0}$$



4 points

HW Set F, Problem 1

Binary Star System



Consider the motion of M_2 — circular motion with radius r_2.

$$\frac{\cancel{M_2} v^2}{r_2} = \frac{G M_1 \cancel{M_2}}{(r_1 + r_2)^2}$$ 2 points

Note that $M_1 r_1 = M_2 r_2$

Distances: $R = r_1 + r_2$ and $M_1 r_1 = M_2 r_2$

Therefore $r_1 = \frac{M_2 R}{M_1 + M_2}$ and $r_2 = \frac{M_1 R}{M_1 + M_2}$

$$v^2 = \frac{G M_1 r_2}{(r_1 + r_2)^2}$$

$$= \frac{G M_1^2 R}{(M_1 + M_2) R^2} = \frac{G M_1^2}{(M_1 + M_2) R}$$

The period of revolution of M_2 (which is equal also to the period of revolution of M_1) is

$$T = \frac{2\pi r_2}{v_2} = \frac{2\pi M_1 R}{M_1 + M_2} \sqrt{\frac{(M_1 + M_2) R}{G M_1^2}}$$

$$\therefore T = \sqrt{\frac{4\pi^2 R^3}{G(M_1 + M_2)}}$$ ← 2 points

$\boxed{\text{4 points}}$

HW Set F, Problem 10

Example 9.12 : Saturn rocket

(a) Velocity v versus time t for the first stage of the rocket, is given by Eq. (9.165)

$$v(t) = -gt + u \ln \frac{m_0}{m_0 - \mu t}$$

PARAMETERS

$g = 9.81$ m/s^2

$u = 2600$ m/s

$m_0 = 2.8 \times 10^6$ kg

μ = exhaust mass rate $= 1.42 \times 10^4$ kg/s

t_B = burnout time $= 148$ s

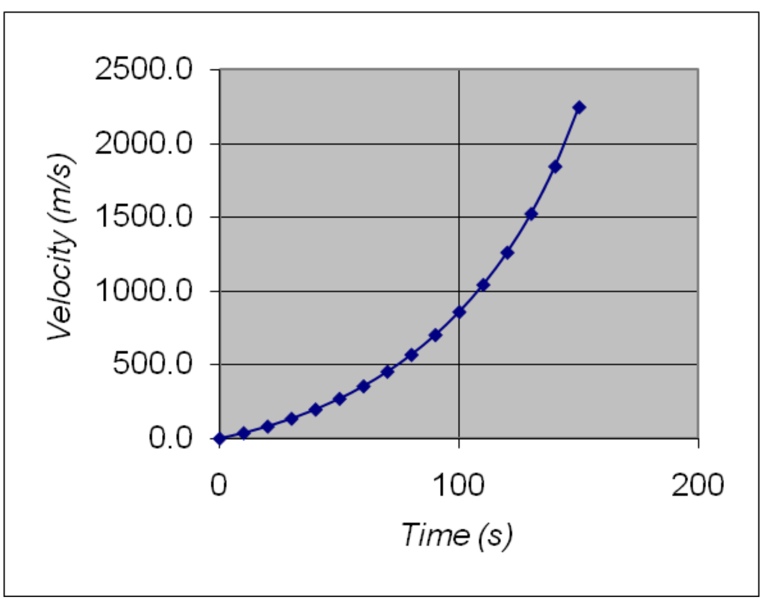
Use EXCEL, or some other computer graphics program, to make an accurate graph of $v(t)$. 2 points

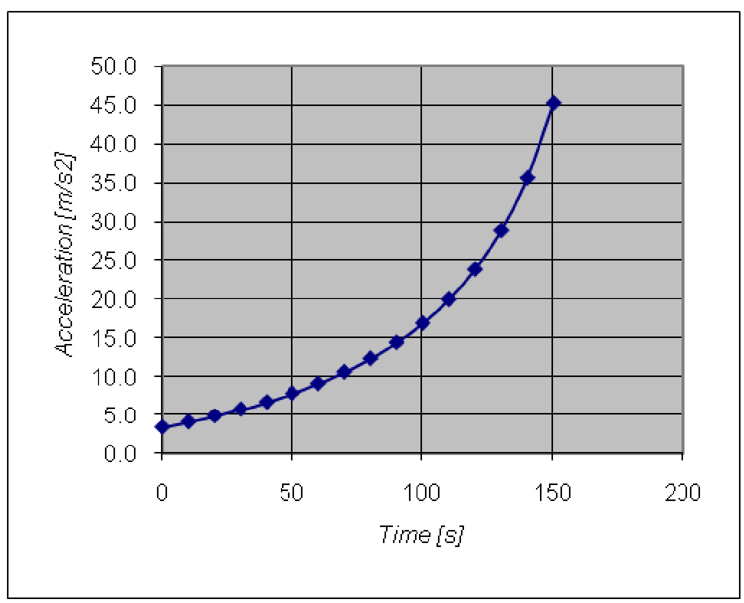
(b) The acceleration is

$$a(t) = \frac{dv}{dt} = -g + \frac{u\mu}{m_0 - \mu t}$$

Use EXCEL, or some other computer program to make an accurate graph of $a(t)$. 2 points

$\boxed{\text{4 points}}$





Linearly damped oscillator (Chap. 3)

$$m\ddot{x} = -\gamma\dot{x} - kx$$

Try $x = e^{\alpha t}$

$\dot{x} = \alpha e^{\alpha t}$

$\ddot{x} = \alpha^2 e^{\alpha t}$

$$m\alpha^2 = -\gamma\alpha - k$$

$$m\alpha^2 + \gamma\alpha + k = 0$$

$$\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Underdamped oscillations have $\gamma^2 < 4mk$.

Then $\alpha = \frac{-\gamma}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$

Define $\omega_0 = \sqrt{\frac{k}{m}}$ and $\omega_d = \sqrt{\omega_0^2 - (\gamma/2m)^2}$

Then $\alpha = -\frac{\gamma}{2m} \pm i\omega_d$

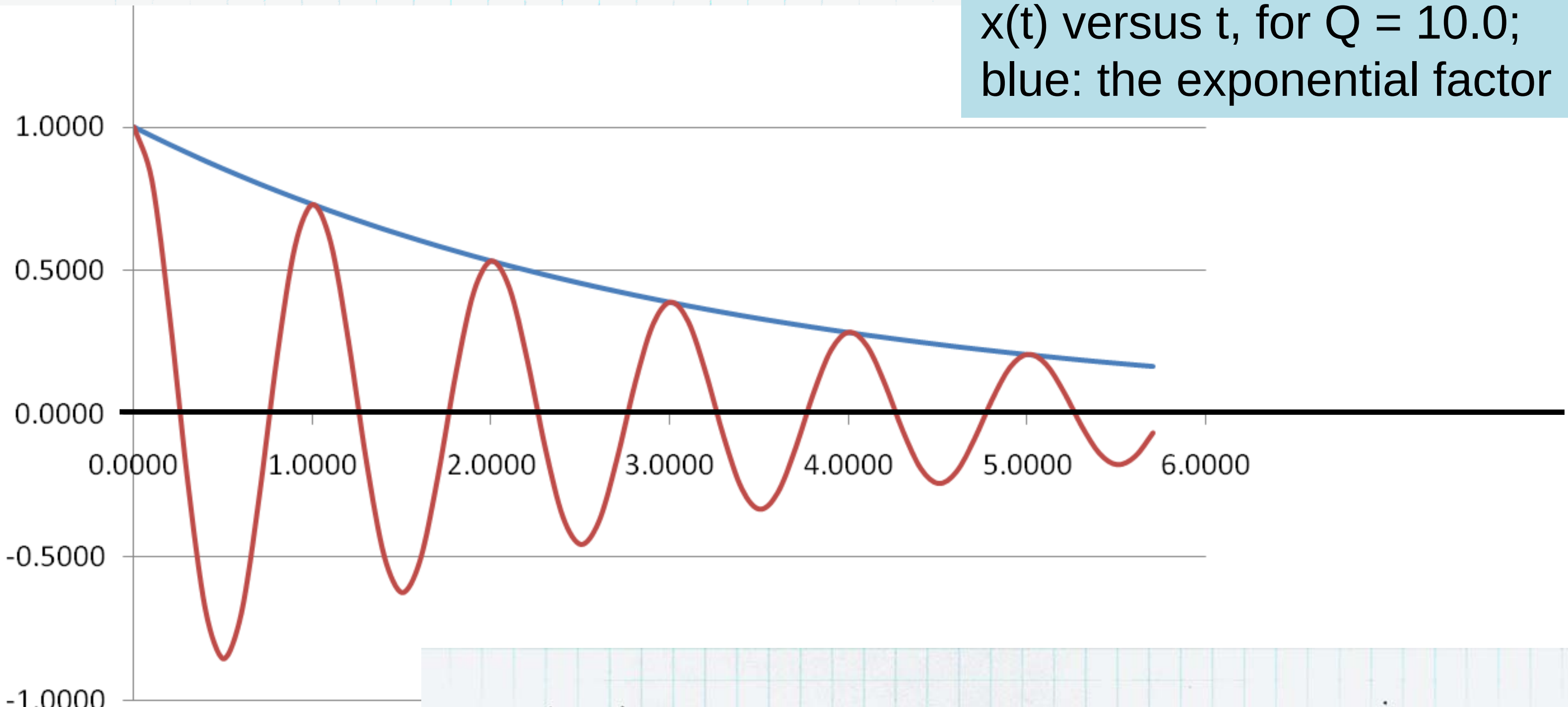
The solution with initial values $\begin{cases} x(0) = A \\ \dot{x}(0) = 0 \end{cases}$ is

$$x(t) = A\, e^{-\gamma t/2m} \left\{ \cos\omega_d t + \frac{\gamma}{2m\omega_d} \sin\omega_d t \right\}$$

Euler $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{\pm i\omega_d t} = \cos\omega_d t \pm i\sin\omega_d t$$

$$x(t) = A\, e^{-\gamma t/2m} \left\{ \cos \omega_d t + \frac{\gamma}{2m\omega_d} \sin \omega_d t \right\}$$



x(t) versus t, for Q = 10.0;
blue: the exponential factor

<u>Velocity</u> After a bit of algebra ...

$$\dot{x}(t) = -A\,\omega_d \left[1 + \left(\frac{\gamma}{2m\omega_d} \right)^2 \right] e^{-\gamma t/2m} \sin \omega_d t$$

Note that $\dot{x}(t) = 0$ when $\omega_d t = \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots$

These are the times of maximum displacement from 0.

$x(t)$ is at a maximum positive displacement for

$$\omega_d t = 0, 2\pi, 4\pi, 6\pi, 8\pi \ldots$$

$$\omega_d t_n = 2\pi n \qquad \text{where} \qquad n = 0, 1, 2, 3, \ldots$$

$$x_n = A\, e^{-\gamma t_n/2m} \{ 1 + 0 \} = A e^{-\frac{2\pi\gamma}{2m\omega_d} n}$$

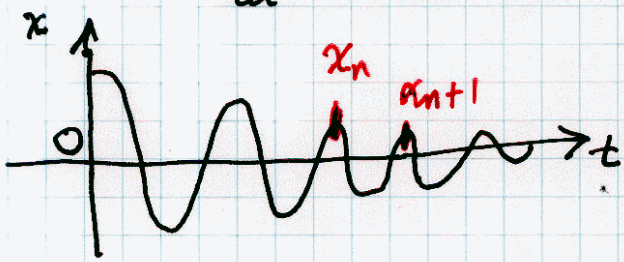
Energy and the underdamped oscillator

Define $E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$.

If $\gamma = 0$ then E is constant.

If γ is "small" then E decreases slowly.

$$\frac{dE}{dt} = m \dot{x} \ddot{x} + k x \dot{x} = (m\ddot{x} + kx)\dot{x} = -\gamma \dot{x}^2$$



$$x_n = A e^{-\frac{\pi \gamma}{m \omega_d} n}$$

$E_n = \frac{1}{2} k x_n^2 =$ the maximum of potential energy for positive displacements.

$$E_n = \frac{1}{2} k A^2 e^{-\frac{2\pi\gamma}{m\omega_d} n} = \frac{1}{2} k A^2 e^{-2\pi n / Q}$$

$$\boxed{\text{Define } Q = \frac{m\omega_d}{\gamma}}$$

"Quality factor" is a dimensionless measure of the damping

Comments

- $$Q = \frac{m\omega_d}{\gamma} \quad \text{where} \quad \omega_d = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$$

  So $$Q = \sqrt{\left(\frac{m\omega_0}{\gamma}\right)^2 - \frac{1}{4}}$$

For weak damping, $Q \approx \frac{m\omega_0}{\gamma}$ and $Q \gg 1$.

- $$\frac{E_n}{E_{n+1}} = \frac{e^{-2\pi n/Q}}{e^{-2\pi(n+1)/Q}} = e^{2\pi/Q} \quad \text{(independent of } n\text{)}$$

  $$Q = \frac{2\pi}{\ln(E_n/E_{n+1})}$$

- $$E_{n+1} = E_n - |\Delta E|_n$$

  $|\Delta E|_n$ = energy lost by friction in one cycle, from n to $n+1$.

  $$\frac{|\Delta E|_n}{E_n} = 1 - \frac{E_{n+1}}{E_n} = 1 - e^{-2\pi/Q}$$

For $Q \gg 2\pi$ $\left\{\text{i.e., } \gamma \ll \frac{m\omega_0}{2\pi}\text{ ; weak damping}\right\}$

$$\frac{|\Delta E|_n}{E_n} \approx \frac{2\pi}{Q} \quad \text{or} \quad \frac{E_n}{|\Delta E|_n} \cong \frac{Q}{2\pi}$$

- Large Q means weak damping ("quality factor")