A mass m moves in one dimension (x) subject to a restoring force $F_R = -kx$ and a damping force $F_D = -\gamma v$.

(a) Show that the equation of motion can be solved with the functional form $x(t) = \exp(\alpha t)$, and derive the equation for α.

(b) The parameter α might be a complex number (i.e., not real) of the form $\alpha = a_1 + i a_2$ with $a_2 \neq 0$. What would that imply?

(c) Derive the condition (relating m, k and γ) such that α is complex.

(d) Does complex α occur for strong or weak damping?

(e) Sketch a graph of $x(t)$ if α is complex.

Solution Key

(a) $m\ddot{x} = -\gamma\dot{x} - kx$

Substitute $x = e^{\alpha t} \Rightarrow m\alpha^2 + \gamma\alpha + k = 0$ 1 pt.

(b) If $\alpha = a_1 + i a_2$ then $x = e^{a_1 t} e^{i a_2 t}$.

This implies an oscillatory function of time. 1 pt.

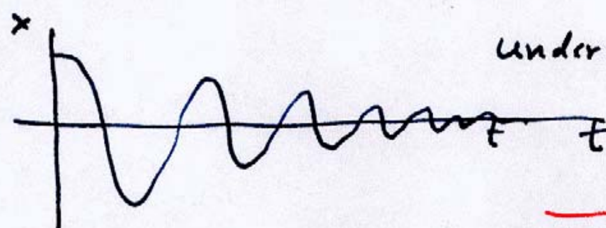
Or, underdamped oscillator.

(c) $$\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

α is complex if $\gamma < \sqrt{4mk}$. 1 pt.

(d) Weak Coupling 1 pt.

(e)

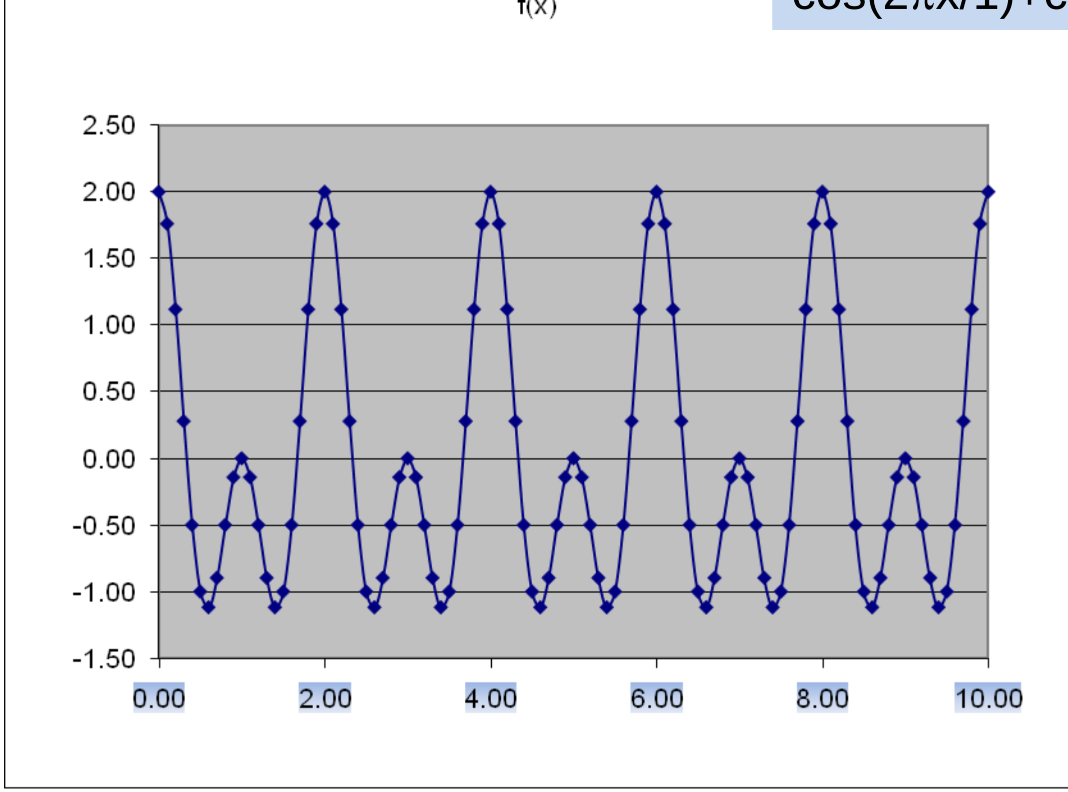


underdamped oscillator.

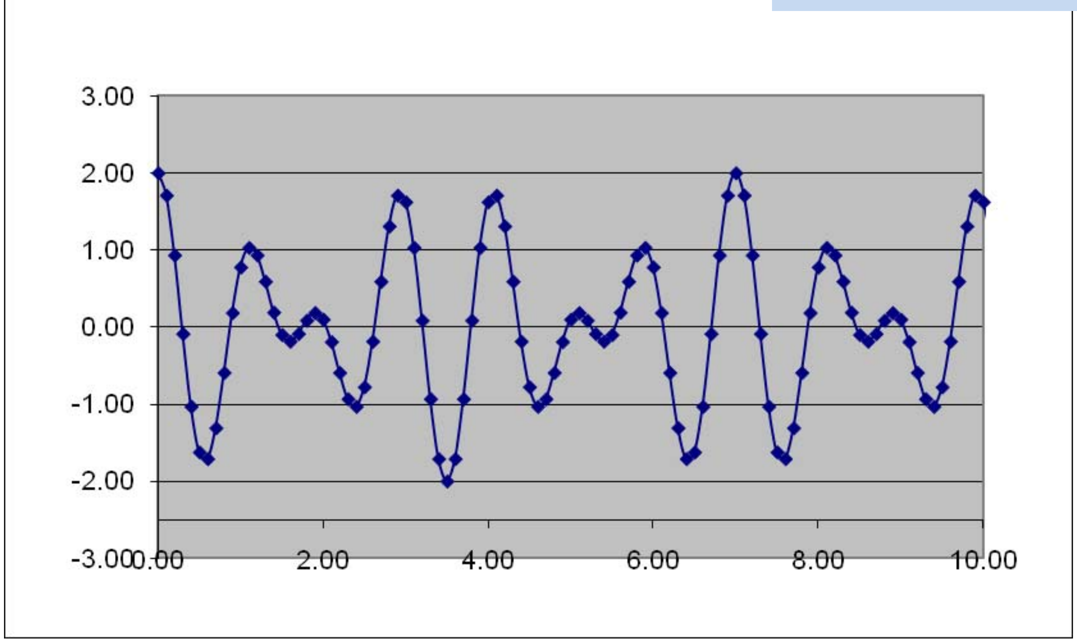
1 pt.

5 points total

cos(2πx/1)+cos(2πx/2)



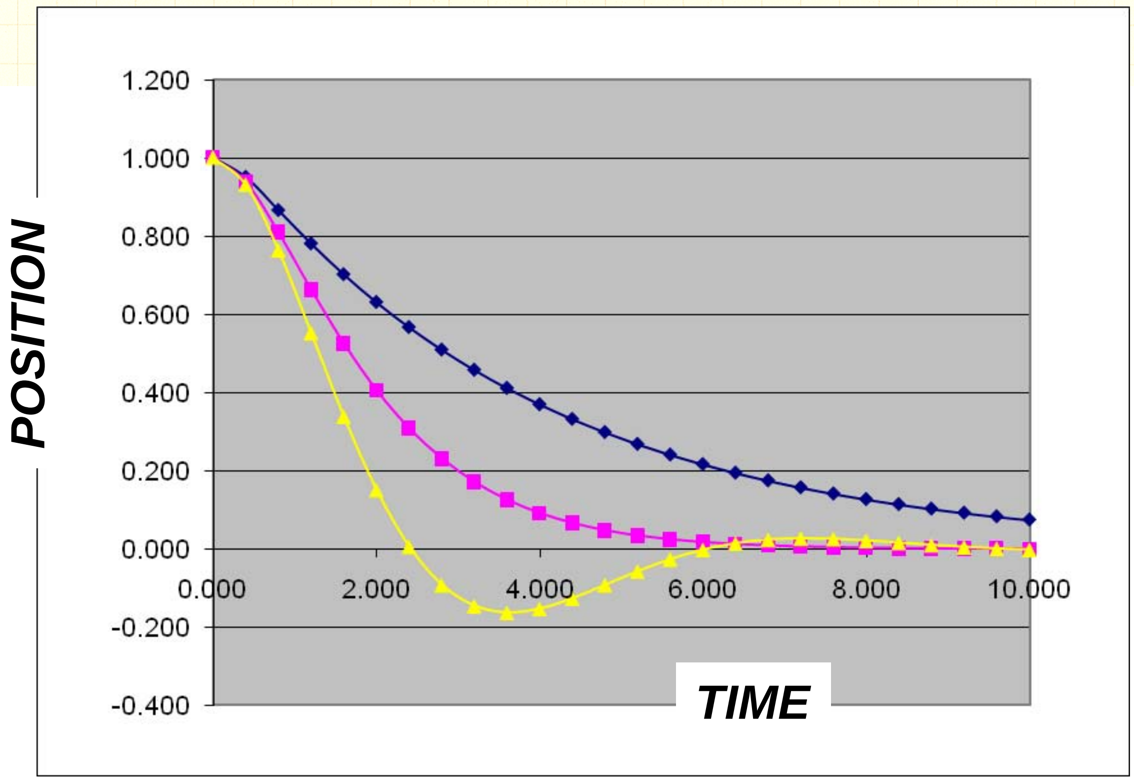
cos(2πx/1)+cos(2πx/1.3)



TM 3-41 (b)

Solve the equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$,
with initial values $x(0) = A = 1$ and $\dot{x}(0) = 0$.

ω_0	β	Solution $x(t)$:	
1	2	$A\,e^{-\beta t}\left\{\cosh \Omega t + \frac{\beta}{\Omega}\sinh \Omega t\right\}$	$\Omega = \sqrt{\beta^2 - \omega_0^2}$
1	1	$A\,e^{-\beta t}\left\{1 + \beta t\right\}$	
1	0.5	$A\,e^{-\beta t}\left\{\cos \omega t + \frac{\beta}{\omega}\sin \omega t\right\}$	$\omega = \sqrt{\omega_0^2 - \beta^2}$



The Driven Oscillator

$$m\ddot{x} = -kx - \gamma\dot{x} + F(t)$$

E.g., $F(t) = F_0 \sin\omega t$ ← harmonic driving force

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\sin\omega t \qquad \begin{cases} 2\beta = \gamma/m \\ \omega_0^2 = k/m \end{cases}$$

The steady-state solution

↑ ($\lim_{t\to\infty}$ for any initial values)

Try $x(t) = A\sin(\omega t - \phi)$

THE STEADY STATE SOLUTION OSCILLATES with the DRIVING FREQUENCY

$$x = A(\cos\phi\sin\omega t - \sin\phi\cos\omega t)$$

$$\dot{x} = A\omega(\cos\phi\cos\omega t + \sin\phi\sin\omega t)$$

$$\ddot{x} = A\omega^2(-\cos\phi\sin\omega t + \sin\phi\cos\omega t)$$

∴ The equation of motion implies

$$\begin{cases} -A\omega^2\cos\phi + 2\beta A\omega\sin\phi + \omega_0^2 A\cos\phi = \frac{F_0}{m} & \leftarrow \boxed{\sin\omega t} \quad (1) \\ A\omega^2\sin\phi + 2\beta A\omega\cos\phi - \omega_0^2 A\sin\phi = 0 & \leftarrow \boxed{\cos\omega t} \quad (2) \end{cases}$$

$$A^2(\omega_0^2 - \omega^2)^2 + 4\beta^2 A^2\omega^2 = \left(\frac{F_0}{m}\right)^2$$ ← Do you see why?

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

Also .... $\tan\phi = \dfrac{2\beta\omega}{\omega_0^2 - \omega^2}$ (by eq. 2)

RESONANCE

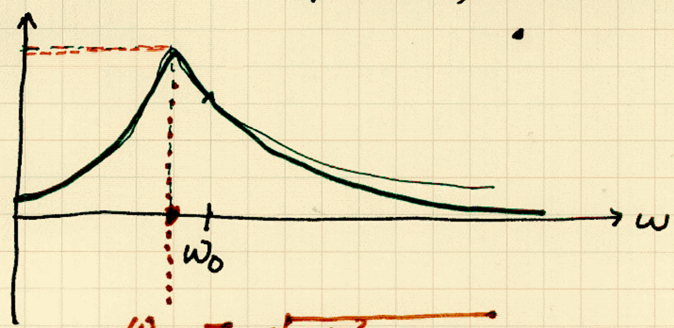
Resonance for small β ; i.e. $\beta \ll \omega_0$

$$x(t) = A \sin(\omega t - \phi)$$

Amplitude $$A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}$$

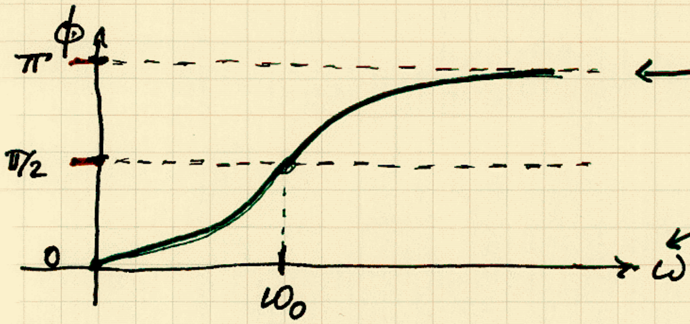
$\dfrac{F_0/m}{2\beta\sqrt{\omega_0^2-\beta^2}} \rightarrow$

$F_0/m\omega_0^2 \rightarrow$



$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$$

Phase shift $$\tan\phi = \frac{2\beta\omega}{\omega_0^2-\omega^2}$$



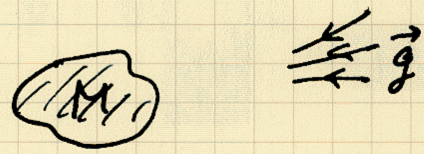
$\leftarrow$ 180° out of phase

$$x(t) = A \sin(\omega t - \pi) = -A\sin(\omega t)$$

$\swarrow$ in phase

Youtube VIDEO of an M.I.T demo.

Chapter 5 — GRAVITATION

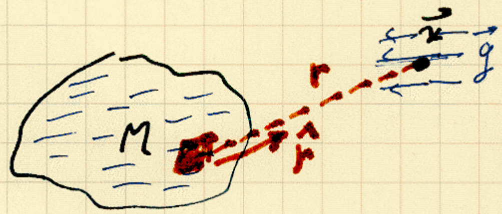


$$m\vec{g} = \vec{F}(\vec{x})$$

$$= -\frac{GMm}{r^2}\,\hat{r} \quad \text{for a sphere.}$$

Gravitational field $\vec{g}(\vec{x}) = \dfrac{\vec{F}(\vec{x})}{m}$ for a test mass m.

$$= -\frac{GM}{r^2}\,\hat{r} \quad \text{for a sphere.}$$



$$\vec{g}(\vec{x}) = \int_V \frac{-G\,dM}{r^2}\,\hat{r}$$ for mass occupying the volume V.

The gravitational potential $\Phi(\vec{x})$, a scalar function

$$\vec{g}(\vec{x}) = -\nabla\Phi$$

$$\Phi(\vec{x}) = \int_V \frac{-G\,dM}{r}$$

/because $\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\,\hat{r}$/

Typical problem:

Given a mass density $\rho(\vec{x}')$, for $\vec{x}' \in V$

determine $\Phi(\vec{x})$ and $g(\vec{x})$. for all $\vec{x}$ in space