Solution Key for Quiz H

$$m\ddot{x} = -kx - \gamma\dot{x} + F_0 \sin\omega t$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\sin\omega t \quad \text{where} \quad 2\beta = \gamma/m, \quad \omega_0^2 = k/m$$

$\boxed{\text{2 points}}$

(a) $x = D\cos\delta\sin\omega t - D\sin\delta\cos\omega t$

$\dot{x} = \omega D\cos\delta\cos\omega t + \omega D\sin\delta\sin\omega t$

$\ddot{x} = -\omega^2 D\cos\delta\sin\omega t + \omega^2 D\sin\delta\cos\omega t$

The equation of motion implies

method: 2 points

Coeff. $\sin\omega t$: $\quad -\omega^2 D\cos\delta + 2\beta\omega D\sin\delta + \omega_0^2 D\cos\delta = F_0/m$

Coeff. $\cos\omega t$: $\quad \omega^2 D\sin\delta + 2\beta\omega D\cos\delta - \omega_0^2 D\sin\delta = 0$

Then $(\omega_0^2 - \omega^2)^2 D^2 + 4\beta^2\omega^2 D^2 = (F_0/m)^2$

answer: 2 points

$$D = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}} \qquad \left\{ \begin{array}{l} \omega_0 = \sqrt{k/m} \\ \beta = \gamma/2m \end{array} \right.$$

(b)

resonance at $\omega_R \approx \omega_0$

graph: 3 points

$\boxed{\text{9 points total}}$

Homework Set H Problem 12 (H1012)

$$m\ddot{x} = -kx - b\dot{x} + F_0\cos\omega t$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \quad \text{where } \beta = b/2m, \quad \omega_0^2 = k/m$$

① Use complex analysis to find the steady-state solution.

Write $\frac{F_0}{m}\cos\omega t = \mathrm{Re}\left\{\frac{F_0}{m}e^{i\omega t}\right\}$ and $x(t) = Ae^{i\omega t}$

(real part at end)

$$(-\omega^2 + 2\beta i\omega + \omega_0^2)A = F_0/m$$

$$A = \frac{F_0/m}{\omega_0^2 - \omega^2 + 2i\beta\omega} = \frac{F_0/m \quad e^{-i\phi}}{\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]^{1/2}} \quad \text{where } \tan\phi = \frac{2\beta\omega}{\omega_0^2-\omega^2}$$

Thus

$$x_{ss}(t) = |A|\cos(\omega t - \phi) \qquad \text{Note: } A = |A|e^{-i\phi}$$

② At resonance, let $\omega = \omega_0$ and $b = m\omega_0$.

For this special case, $\beta = \frac{b}{2m} = \frac{m\omega_0}{2m} = \frac{\omega_0}{2}$

Also $\tan\phi = \infty$ so $\phi = \pi/2$.

Thus $x_{ss} = |A|\sin\omega_0 t$ and $|A| = \frac{F_0/m}{2\beta\omega_0} = \frac{F_0}{m\omega_0^2}$

③ Transient Solution

Consider these initial values: $x(0) = 0$ and $v(0) = 0$.

Write

$$x(t) = \underbrace{e^{-\beta t}\left\{C\cos\omega_1 t + D\sin\omega_1 t\right\}}_{\text{solves the homogeneous equation}} + \underbrace{|A|\sin\omega_0 t}_{\text{steady state solution}}$$

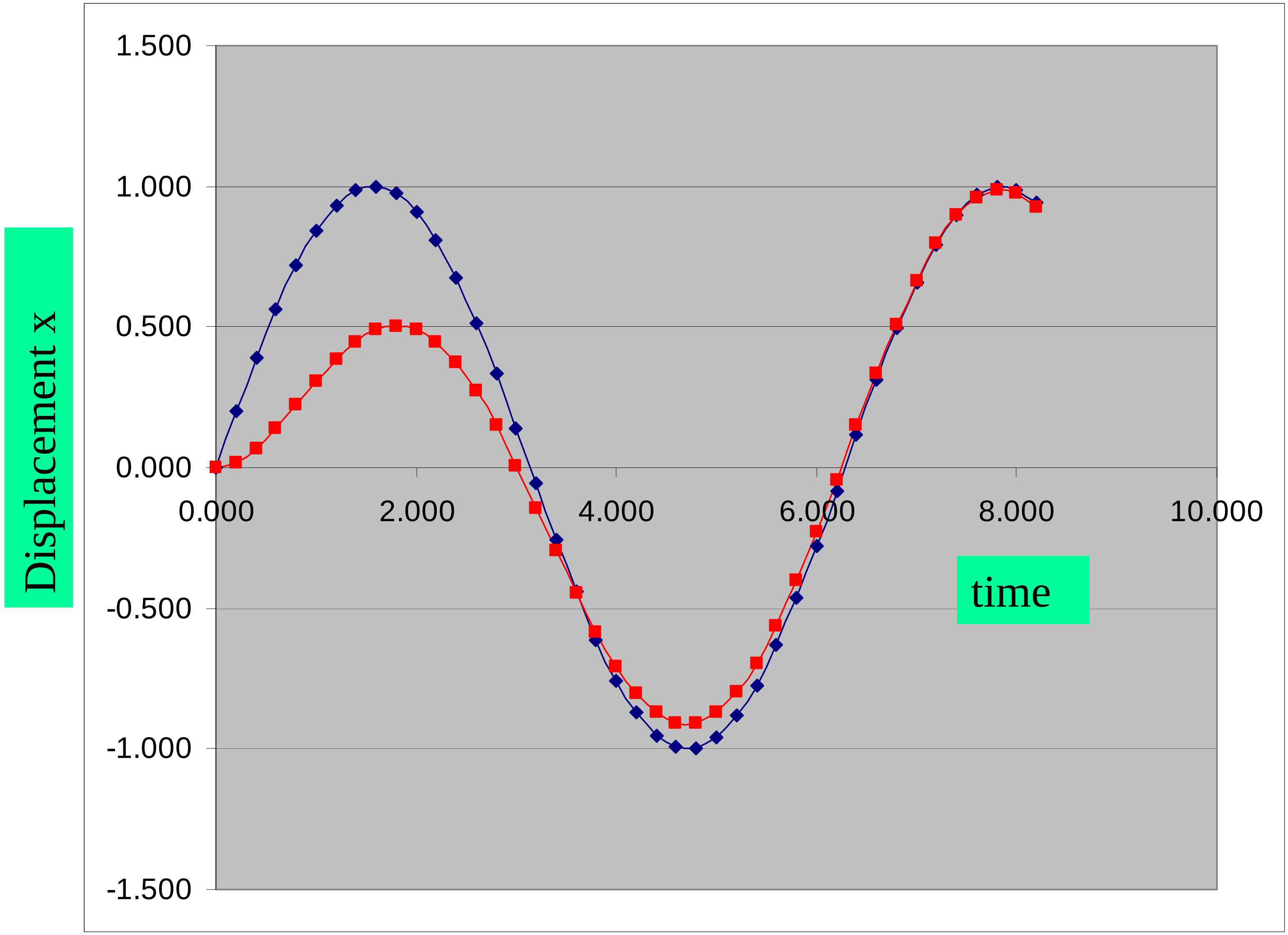
where $\omega_1 = \sqrt{\omega_0^2 - \beta^2} = \frac{\sqrt{3}}{2}\omega_0$.

Initial values:

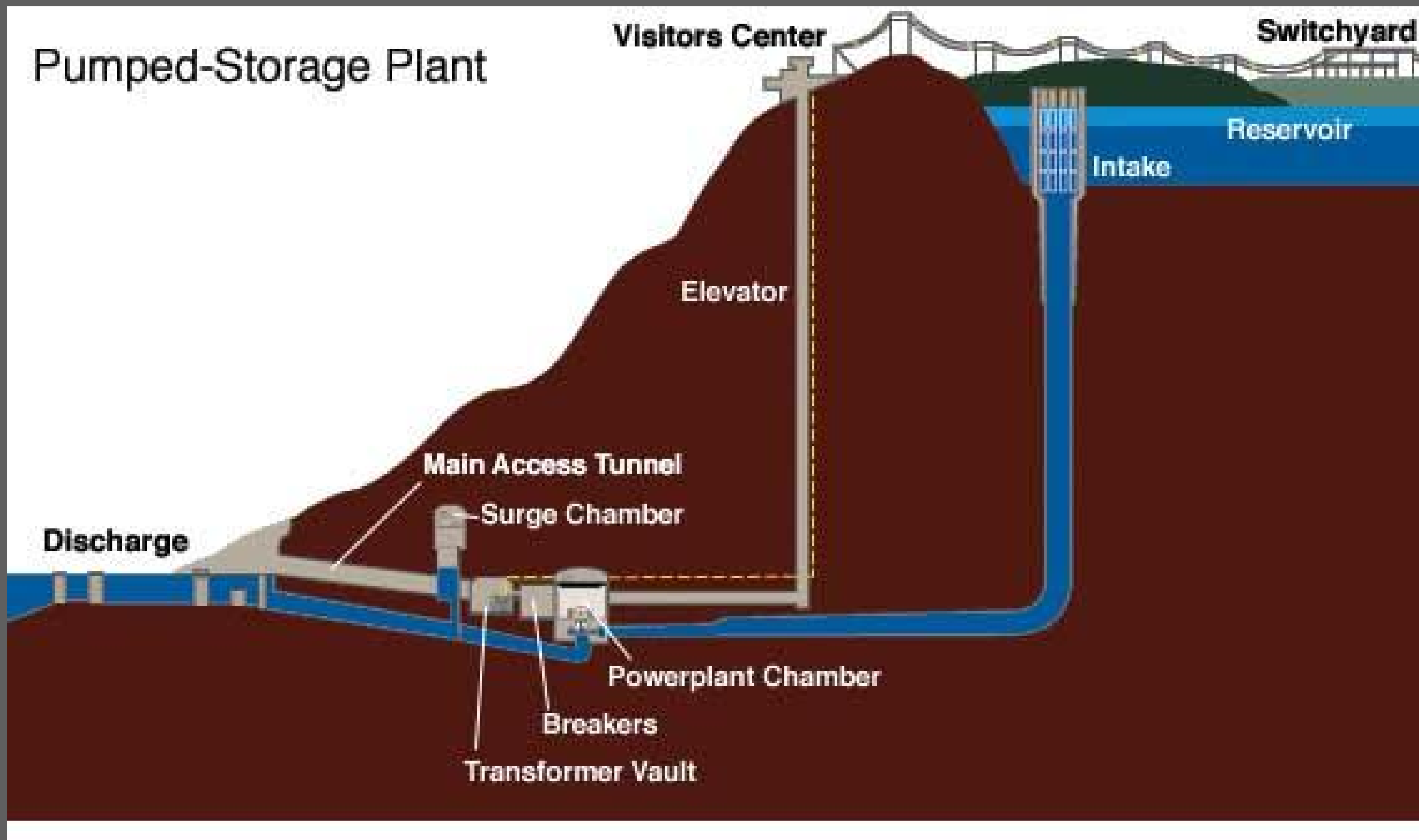
$x(0) = 0 \Rightarrow C = 0$ and $v(0) = 0 \Rightarrow D = \frac{-\omega_0}{\omega_1}|A|$

$$\boxed{x(t) = |A|\left\{\sin\omega_0 t - \frac{\omega_0}{\omega_1}e^{-\beta t}\sin\omega_1 t\right\} \quad \text{where } |A| = \frac{F_0}{m\omega_0^2}\ ;\ \beta = \frac{1}{2}\omega_0\ ;\ \omega_1 = \frac{\sqrt{3}}{2}\omega_0.}$$

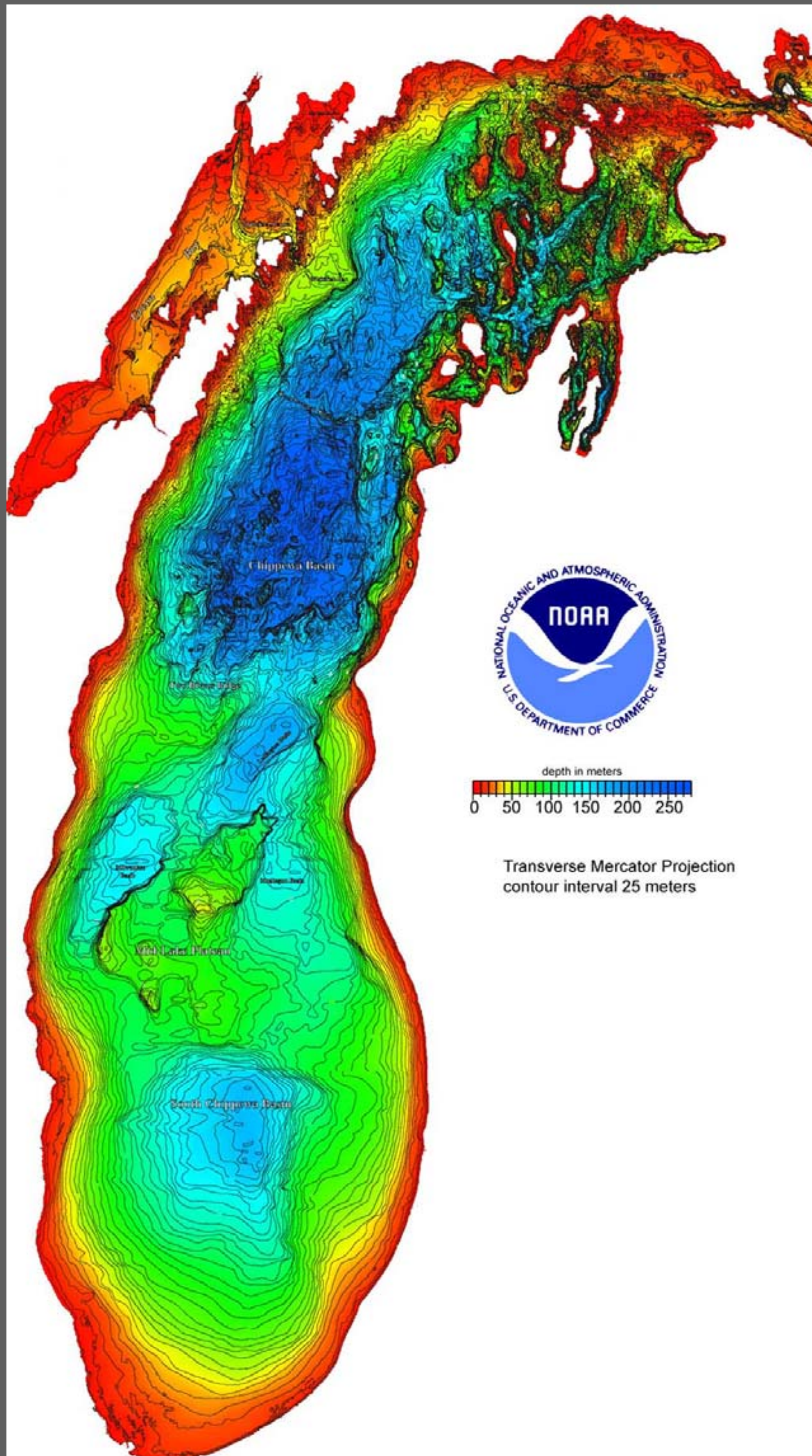
Homework Set H Problem 12 (H1012)



4 points maximum

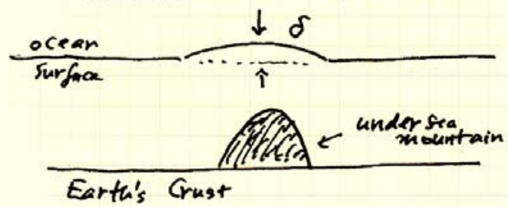

Pumped-Storage Plant
Visitors Center
Switchyard
Reservoir
Intake
Elevator
Main Access Tunnel
Surge Chamber
Discharge
Powerplant Chamber
Breakers
Transformer Vault

The world's largest hydro-storage facility al Ludington, Michigan, uses Lake Michigan as the lower reservoir and an artificial lake 100 m higher as the upper reservoir. This plant can deliver 2000 MW at full power and can store 15,000 MWh of energy



http://www.ngdc.noaa.gov/mgg/bathymetry/predicted/explore.HTML

Satellite Bathymetry



Parameters		e.g.
ocean depth	D	4000 m
Mountain height	H	2000 m
" radius	R	15 km
" density	ρ	3×10^3 kg/m^3

Paraboloid $r_\perp = \sqrt{1 - z/H}\, R$

The ocean surface is a gravitational equipotential.
Calculate δ = displacement of the surface above the mountain.

Far from the mountain ($\sqrt{x^2+y^2} \gg R$)

$$\Phi(x,y,z) = \frac{U}{m} = \frac{mgz}{m} = gz$$

$$\Phi(x,y,D) = gD$$

Above the mountain ($x=0$ and $y=0$)

$$\Phi(0,0,D+\delta) = g(D+\delta) + \Phi_{mountain}$$

It's an equipotential, so $\Phi(x,y,D) = \Phi(0,0,D+\delta)$

$$\therefore\ \delta = -\frac{\Phi_{mountain}}{g} \quad \text{where } \Phi_{mountain} = -\int_{mountain} \frac{G\,dm}{r}$$

Approximation and Bound

$$\delta = \frac{R_\oplus^2}{GM_\oplus}\int_V \frac{G\,dm}{r} < \frac{R_\oplus^2}{GM_\oplus}\,\frac{G}{D-H+\delta}\, m$$

$$\boxed{r > D-H+\delta}$$

MASS

$$m = \rho\int dV = \rho\int_0^H dz\,(\pi r_\perp^2)$$

$$= \rho\pi R^2\int_0^H \left(1-\frac{z}{H}\right)dz = \rho\frac{\pi}{2}R^2 H$$

$$\delta < \frac{R_\oplus^2}{M_\oplus}\,\frac{\rho\frac{\pi}{2}R^2H}{D-H}$$

Plug in the numbers $\Rightarrow$ $\delta < 7$ m.

FINAL EXAM : Calculate the integral accurately.