Physics 472 – Spring 2009

Homework #9, due Friday, March 27

(Point values are in parentheses.)

1. [7] The isotropic 2-dimensional harmonic oscillator is easily solved by writing the Hamiltonian as a sum of x and y Hamiltonians:

$$\hat{H}^0 = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m} + \tfrac{1}{2}m\omega^2\left(x^2 + y^2\right) = \hat{H}_x^0 + \hat{H}_y^0 \quad \text{with } \left[\hat{H}_x^0, \hat{H}_y^0\right] = 0\,.$$

Simultaneous eigenstates of $\hat{H}_x^0$ and $\hat{H}_y^0$ obey $\hat{H}^0\left|n_x, n_y\right\rangle = \left(n_x + n_y + 1\right)\hbar\omega\left|n_x, n_y\right\rangle$.

a) Consider the perturbation $\lambda\hat{H}' = \lambda m\omega^2 xy$. Calculate the first and second order energy shifts of the ground state. In class we used $\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_x^+ + \hat{a}_x\right)$ to evaluate the matrix elements:

$\left\langle n_x'\right|x\left|n_x\right\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n_x}\,\delta_{n_x',n_x-1} + \sqrt{n_x+1}\,\delta_{n_x',n_x+1}\right)$. The same holds for $\hat{Y} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_y^+ + \hat{a}_y\right)$.

To evaluate the matrix elements of $\hat{H}'$ in the $\left|n_x n_y\right\rangle$ tensor product states, use:

$$\left\langle n_x' n_y'\right|xy\left|n_x n_y\right\rangle = \left\langle n_x'\right|x\left|n_x\right\rangle\left\langle n_y'\right|y\left|n_y\right\rangle$$

b) Use degenerate P.T. to calculate the first-order energy shifts of the first excited states, as well as the "correct" linear combinations of those two states that diagonalize $\hat{H}'$.

c) The full Hamiltonian, $\hat{H} = \hat{H}^0 + \lambda\hat{H}'$, is exactly solvable if you make the coordinate transformation $u = (x+y)/\sqrt{2}$, $v = (x-y)/\sqrt{2}$. Express $\hat{H}$ in terms of *u*, *v*, and their conjugate momenta P_u, and P_v. You should find that the harmonic oscillator in the "*u*" direction has a higher frequency than before, while in the "*v*" direction the frequency is lower. Calculate the exact energies of the new basis states $\left|n_u, n_v\right\rangle$. For the ground state, expand the energy to second order in λ. For the next two higher states, expand the energies to first order in λ. Compare your results with those you obtained in parts (a) and (b).

2. [7] Consider an electron in a 3-dimensional isotropic harmonic oscillator potential, in the presence of a uniform magnetic field $\vec{B} = B_{ext}\hat{k}$. The full Hamiltonian for the system is:

$$\hat{H} = \frac{\hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2}{2m} + \tfrac{1}{2}m\omega^2\left(x^2 + y^2 + z^2\right) + \frac{eB_{ext}}{2m}\left(L_z + 2S_z\right)$$

This problem is exactly solvable, but we'll use the machinery of perturbation theory to get organized. Treat the first two terms of H as H^0, and the Zeeman term as $\lambda H'$. The eigenstates of H^0 satisfy: $\hat{H}^0\left|n_x, n_y, n_z, m_s\right\rangle = \left(n_x + n_y + n_z + \tfrac{3}{2}\right)\hbar\omega\left|n_x, n_y, n_z, m_s\right\rangle$, where $m_s = \pm\tfrac{1}{2}$, and $S_z\left|n_x, n_y, n_z, m_s\right\rangle = \hbar m_s\left|n_x, n_y, n_z, m_s\right\rangle$. (I am putting the space and spin quantum numbers together inside the same ket to avoid using the cumbersome tensor product notation.)

a) Express $\hat{L}_z$ in terms of the harmonic oscillator raising and lowering operators. Hint: you should get $\hat{L}_z = i\hbar\left(a_x a_y^+ - a_x^+ a_y\right)$.

b) The ground state of H^0 is 2-fold degenerate, due to spin. But since the two $|0,0,0,m_s\rangle$ states are already eigenstates of H', you can use standard first-order perturbation theory to calculate the energy shifts due to the magnetic field. Express your answers in terms of $\mu_B B_{ext}$ and m_s.

c) The first excited state of H^0 is six-fold degenerate (3 spatial states $\times$ 2 spin states). Calculate the linear combinations of states that diagonalize H'. To help you keep track of what you are doing, here are some suggestions. First, since all your states are eigenstates of S_z, leave spin out of the problem until the end; then you only have to diagonalize a 3×3 matrix rather than a 6×6 matrix. Since the original basis states $|n_x, n_y, n_z, m_s\rangle$ are not eigenstates of $\hat{L}_z$, you need to find linear combinations of them that are. Label the new states this way: $|n, l, m_l, m_s\rangle$, where $n = n_x + n_y + n_z$. You don't need to know l to do this problem – you just need m_l. But you can probably guess what l is once you know what m_l is for the three states. Finally, when you have found the states that diagonalize H', calculate the Zeeman energy shifts of those states. How many distinct energies are there? Make a plot of energy vs. $\mu_B B_{ext}$ for all the states.

d) The second excited state of H^0 is twelve-fold degenerate (6 spatial states $\times$ 2 spin states). Forget about spin altogether so you don't get lost. Construct the 6×6 matrix representation of $\hat{L}_z$. If you choose the order of your 6 states judiciously, your 6×6 matrix should break up into a 2×2 block, a 3×3 block, and a 1×1 block. Calculate the eigenvalues of $\hat{L}_z$ and their degeneracies. Guess what the values of l are for this six-dimensional subspace. Don't bother to calculate the 12 Zeeman energies – I know you could do it if you had to!

3. [6] Griffiths problem 6.37. Follow the same strategy you used to solve Griffiths problem 6.36. Use symmetries to figure out which matrix elements of the form $\langle n, l', m_l'|z|n, l, m_l\rangle$ are zero. Rotational symmetry, $\left[\hat{z}, \hat{L}_z\right] = 0$, implies $(m_l' - m_l)\langle n, l', m_l'|z|n, l, m_l\rangle = 0$. The parity transformation, $\hat{\Pi}\hat{z}\hat{\Pi} = -\hat{z}$, implies $(-1)^{l'+l}\langle n, l', m_l'|z|n, l, m_l\rangle = -\langle n, l', m_l'|z|n, l, m_l\rangle$.

Use these same symmetries to do Griffiths problem 6.37. First, show which elements of the 9×9 matrix are zero. Then calculate the first non-zero matrix element, $\langle 3,0,0|z|3,1,0\rangle$, using Tables 4.3 and 4.7 in Griffiths. Use Mathematica to do the radial integration. You can take the values of the other nonzero matrix elements from Griffiths. Construct the 9×9 matrix representation of $\hat{z}$. If you choose the order of the 9 states carefully, then the matrix should break into a 3×3 block, two 2×2 blocks, and two trivial 1×1 blocks. Calculate the eigenvalues and their degeneracies. Don't forget to multiply the eigenvalues by eE_{ext} to get the energies.