

Physics 472 Midterm Exam #1 -- Monday, February 16, 2009

Total points = 20. Show all your work!

Useful relations about Angular Momentum:

$$[J_x, J_y] = i\hbar J_z \text{ and cyclic permutations.}$$

$$J^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$$

$$J_z |j, m\rangle = \hbar m |j, m\rangle$$

$$J_+ = J_x + iJ_y$$

$$J_- = J_x - iJ_y$$

$$J_+ |j, m\rangle = \hbar \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle$$

$$J_- |j, m\rangle = \hbar \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle$$

Spin-1/2:

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}, \quad \text{where } \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over every coefficient, e.g., for $-8/15$ read $-\sqrt{8/15}$.

Notation: $\begin{matrix} J & J & \dots \\ M & M & \dots \end{matrix}$

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