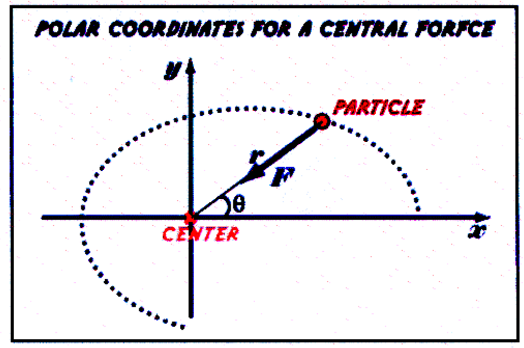
PHY 321 Quiz K August 4, 2009

A particle of mass m moves under the influence of a central force; $F = -\nabla V$ where $V(r)$ is the potential energy. Angular momentum is conserved because the force is central. The constants of the motion are L and energy E.

We use polar coordinates $r(t)$ and $\theta(t)$ to describe the motion. ***Derive an equation for*** $dr/d\theta$ ***as a function of r.*** [The solution of this first-order differential equation determines the orbit.]



$\boxed{\text{SOLUTION KEY}}$

$$E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} + V(r) \quad \text{1 pt} \qquad \left(\text{Note: } \frac{L^2}{2mr^2} = \frac{1}{2} m r^2 \dot{\theta}^2\right)$$

$$L = m r^2 \dot{\theta} \quad \text{1 pt}$$

$$d\theta = \frac{L}{mr^2}\, dt$$

$$dr = \pm \sqrt{\frac{2}{m}(E - V_{eff}(r))}\, dt \quad \text{where} \quad V_{eff}(r) = V(r) + \frac{L^2}{2mr^2}$$

method 1 pt

$$\frac{dr}{d\theta} = \pm \frac{mr^2}{L} \sqrt{\frac{2}{m}(E - V_{eff})}$$

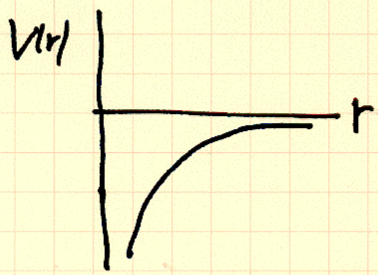
answer 1 pt

$\boxed{\text{5 pts}}$

$$\frac{dr}{d\theta} = \pm \frac{mr^2}{L} \sqrt{\frac{2}{m}(E - V_{eff}(r))} \quad \text{where} \quad V_{eff}(r) = V(r) + \frac{L^2}{2mr^2}$$

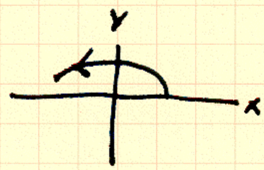
Problem K904 (Handed in with H.W. Set K)

Let $V(r) = \frac{-A}{r^n}$ where $A > 0$.



The force is directed toward $r = 0$.

For what values of n will the circular orbits be stable?



Orbit Equations

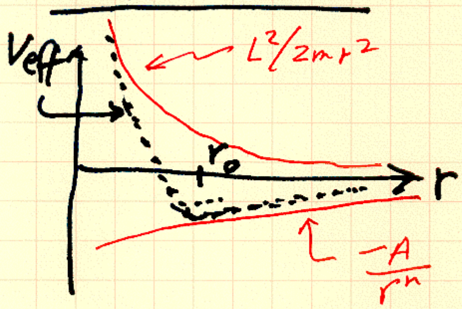
$L = mr^2 \dot{\phi} = \text{const.}$

$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r^2\dot{\phi}^2) + V(r) = \text{const.}$

$= \frac{1}{2}m\dot{r}^2 + \underbrace{V(r) + \frac{L^2}{2mr^2}}$

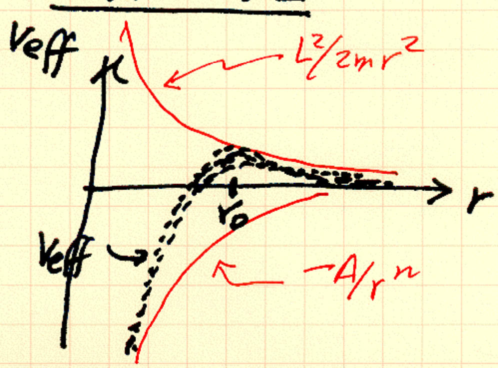
$= \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$

Case $n < 2$



The circular orbit $(r = r_0)$ is stable.

Case $n > 2$

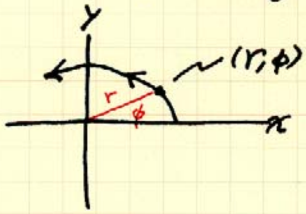


The circular orbit $(r = r_0)$ is unstable.

A The circular orbits are stable if $n < 2$.

Problem K906 (Handed in with HW Set K)

A particle of mass m moves under the influence of potential energy $V(r) = \frac{1}{2}kr^2$.

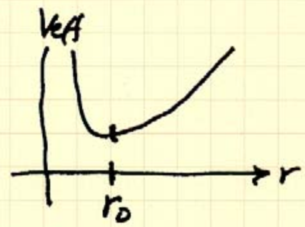


$$L = mr^2\dot{\phi} = \text{constant}$$

$$E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r) = \text{const.}$$

$$V_{eff}(r) = V(r) + \frac{L^2}{2mr^2}$$

$$V_{eff}(r) = \frac{1}{2}kr^2 + \frac{L^2}{2mr^2}$$



The circular orbit has $r(t) = r_0$ where $\frac{dV_{eff}}{dr} = 0$;

i.e., $k r_0 - \frac{L^2}{m r_0^3} = 0$

$$r_0 = \left(\frac{L^2}{mk}\right)^{1/4} \quad \text{and} \quad E = V_{eff}(r_0) = \left(\frac{kL^2}{m}\right)^{1/2}.$$

Now consider small oscillations around the circular orbit.

Write $r(t) = r_0 + \epsilon(t)$

$$V_{eff}(r) \approx V_{eff}(r_0) + \frac{1}{2}\epsilon^2\left(\frac{d^2V_{eff}}{dr^2}\right) = C + \frac{1}{2}K\epsilon^2$$

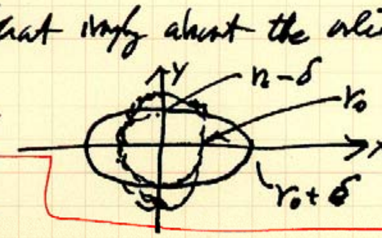
Here $K = V_{eff}'' = k + \frac{3L^2}{mr_0^4} = 4k.$

The radial frequency is $\omega_r = \sqrt{\frac{K}{m}} = 2\sqrt{\frac{k}{m}}$ //HARMONIC RADIAL OSCILLATIONS//

The orbital frequency is $\omega_\phi = \dot{\phi} = \frac{L}{mr_0^2} = \frac{L}{m}\left(\frac{mk}{L^2}\right)^{1/2} = \sqrt{\frac{k}{m}}$

$\boxed{\text{Result} \quad \omega_r = 2\omega_\phi}$ What does that imply about the orbit?

The orbit is approximately an ellipse



IN FACT, the orbits are exactly ellipses. Do you see why?
$m\ddot{x} = -kx$ and $m\ddot{y} = -ky$ in Cartesian coordinates.

Quiz L (8/11/09)

<u>Keplerian Orbits</u>

To determine the orbits, we use the constants of the motion. There are two physical parameters: m = the mass of the planet; and K = the force parameter = GMm.

$$L = m r^2 \dot{\theta}$$
$$E = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} - \frac{K}{r}$$

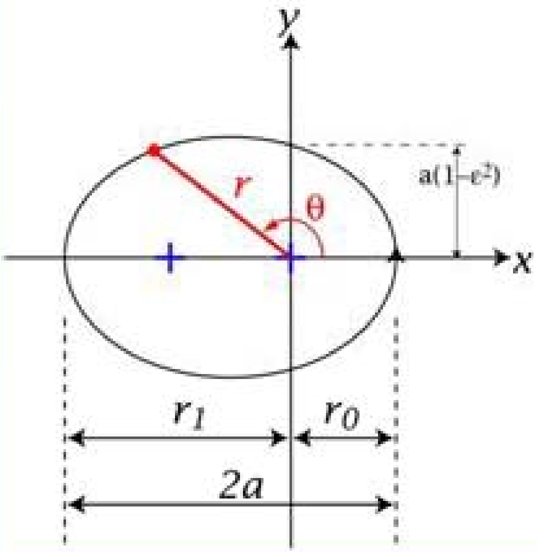
Now derive a differential equation for the distance r versus angle θ. Then it can be shown that the solution for r as a function of θ is

$$\frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \frac{mr^2}{L}\sqrt{\frac{2}{m}\left(E - \frac{L^2}{2mr^2} + \frac{K}{r}\right)}$$

$$\frac{1}{r} = \frac{mK}{L^2} + A\cos\theta \quad \text{where } A^2 = \frac{2mE}{L^2} + \left(\frac{mK}{L^2}\right)^2$$

This is the equation for an ellipse, with one focal point at r = 0.

Ellipse Geometry



r_0 = perihelion distance (= r at $\theta = 0$)

r_1 = aphelion distance (= r at $\theta = \pi$)

a = semi-major axis = $(r_1 + r_0)/2$

ε = eccentricity = $(r_1 - r_0)/(r_1 + r_0)$

$r_0 = a(1-\varepsilon)$ and $r_1 = a(1+\varepsilon)$

The equation for the ellipse is

$$r(\theta) = \frac{a(1-\varepsilon^2)}{1 + \varepsilon\cos\theta}$$

(A) Express L as a function of (a,ε) {and the parameters m, K}

(B) Express E as a function of (a,ε) {and the parameters m, K}.