Physics 472 – Spring 2009

Homework #1, due Friday, January 16

[Point values are in square brackets.]

1. [15] Redo the entire final exam from Physics 471.

2. [5] In Physics 471, you learned three different languages to describe quantum mechanics. They are: 1) differential operators acting on wavefunctions; 2) matrices acting on column vectors; and 3) abstract operators acting on "kets." These are all equivalent, but it's not always easy to see how. Let's see how this works in the case of the 1D harmonic oscillator, with Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2} m\omega^2 \hat{x}^2$$

In the position representation, $\hat{p} \to \frac{\hbar}{i}\frac{\partial}{\partial x}$, so the Hamiltonian operator becomes

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2} m\omega^2 x^2 .$$

In principle, we can solve the time-independent Schrodinger equation to find the energy eigenstates and energies. But we know that there is an easier way, using the ladder operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i\hat{p}}{\sqrt{2\hbar m\omega}}, \quad \hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i\hat{p}}{\sqrt{2\hbar m\omega}}, \text{ and } \hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right).$$

Let's label the energy eigenstates in the usual way:

$$\hat{H}|n\rangle = E_n|n\rangle, \text{ where } E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega, \text{ for } n = 0, 1, 2, \ldots$$

We know what the ladder operators do to the states:

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \text{ and } \hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle .$$

Now let's represent the states as column vectors, using the energy eigenstates as our basis:

$$|0\rangle \to \begin{pmatrix}1\\0\\0\\\ldots\end{pmatrix} \qquad |1\rangle \to \begin{pmatrix}0\\1\\0\\\ldots\end{pmatrix} \qquad |2\rangle \to \begin{pmatrix}0\\0\\1\\\ldots\end{pmatrix} \qquad \text{etc.}$$

a) Write down the first five rows and columns of the matrices representing $\hat{a}$ and $\hat{a}^+$. To get you started, recall that $\hat{a}_{mn} \equiv \langle m|\hat{a}|n\rangle = \sqrt{n}\langle m|n-1\rangle = \sqrt{n}\,\delta_{m,n-1}$.

b) Now write down the first five rows and columns of the matrices representing $\hat{x}$ and $\hat{p}$. (I suggest you take a constant factor outside, so the insides of the matrices just contain numbers.)

c) Now write down the first five rows and columns of the matrices representing $\hat{x}^2$ and $\hat{p}^2$.

d) Finally, write down the first five rows and columns of the Hamiltonian matrix, from the equation $\hat{H} = \frac{\hat{p}^2}{2m} + \tfrac{1}{2} m\omega^2 \hat{x}^2$. If you did everything correctly, this matrix should have a very simple form!