

Physics 472 – Spring 2009

Homework #5, due Friday, February 13

(Point values are in parentheses.)

1. [8] On a previous problem set you found the matrices that represent the operators for the three component of spin, for a spin-1 particle. They are:

$$\hat{S}_x = \begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{pmatrix}, \hat{S}_y = \begin{pmatrix} 0 & -i\hbar/\sqrt{2} & 0 \\ i\hbar/\sqrt{2} & 0 & -i\hbar/\sqrt{2} \\ 0 & i\hbar/\sqrt{2} & 0 \end{pmatrix}, \hat{S}_z = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}$$

An SG apparatus is used to prepare a beam of $s=1$ atoms in the initial state: $|\chi\rangle = \begin{pmatrix} -3i/4 \\ \sqrt{6}/4 \\ i/4 \end{pmatrix}$.

a) [1] If you send the beam into a second SG apparatus with its spin axis oriented along the z -direction, calculate the probabilities associated with each of the three output ports of the apparatus, i.e. calculate the probabilities that measurement of S_z will produce \hbar , 0 , and $-\hbar$. (You should be able to do this just by looking at $|\chi\rangle$, without any calculation.)

b) [2] Now change the spin orientation of your second SG apparatus to the x -axis, and calculate the probabilities associated with each of the three output ports. This time you will need to do a real calculation. You may use the S_x eigenstates you calculated on the PHY471 final exam.

c) [2] Do the same thing with the second SG apparatus oriented along the y -axis.

d) [2] Calculate the expectation values, $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$, using the probabilities you calculated in parts (a) – (c). Check your answers using direct matrix multiplication.

e) [1] Expectation values of spin operators behave like vectors. The state $|\chi\rangle$ given above has the maximum value $\langle S_{\theta,\phi} \rangle = +\hbar$ along an unknown direction labeled by θ , ϕ , and therefore has the following expectation values along the x , y , and z axes:

$$\langle S_x \rangle = \hbar \sin \theta \cos \phi \quad \langle S_y \rangle = \hbar \sin \theta \sin \phi \quad \langle S_z \rangle = \hbar \cos \theta.$$

Find the values of θ and ϕ for which your results from part (d) satisfy these three equations.

2. [4] Griffiths 5.12. Part (a) is trivial. We did some of part (b) in class, but I want you to do it to make sure you understand the logic. For each element, first write down the possible values for the total spin s and total orbital angular momentum l , then consider which values can go together consistently with the Spin-Statistics Theorem. (This is very easy until you get to carbon.) Finally, write down the possible values of j for each combination of s and l , then put the whole thing in spectroscopic notation. Don't do nitrogen unless you want a real challenge! Treat it as extra credit.

3. [4] Griffiths 5.13. When you get to part (d), note that I already told you to consider the Spin-Statistics Theorem in the previous problem. But now you should be able to figure out the correct electronic configuration for nitrogen, even if you were clueless when you did problem 5.12.
4. [4] Consider two neutrons in free space. From classical mechanics, you know that we can specify their motion using the center-of-mass coordinate $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ and the relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$. (To remind yourself how this works, look at Griffiths problem 5.1). For this problem, we'll assume that the center-of-mass momentum is zero, so we'll ignore \vec{R} and \vec{P} . We could thus write the wavefunction for the two-neutron system in the following form:

$$\Psi_{nlms_1s_2}(r, \theta, \phi) = \Phi_{nl}(r) Y_l^m(\theta, \phi) \chi(s_1, s_2).$$

where the radial wavefunction depends both on the quantum number n and the orbital angular momentum l . Instead, let's use eigenstates of total spin S^2 and S_z , where $\vec{S} = \vec{S}_1 + \vec{S}_2$. In Dirac notation, a complete basis set of two-neutron quantum states is thus:

$$|n, l, m_l, s, m_s\rangle \equiv |n, l, m_l\rangle \otimes |s, m_s\rangle$$

When the spatial state is written in terms of the relative coordinate \vec{r} , spatial exchange of the two particles corresponds to the parity transformation, $\vec{r} \rightarrow -\vec{r}$. In terms of spherical polar coordinates, that corresponds to $r \rightarrow r$, $\theta \rightarrow \pi - \theta$, $\phi \rightarrow \phi + \pi$. If you look at the spherical harmonics in Table 4.3, you'll see that they have even parity if l is even, and odd parity if l is odd. In operator notation, that means $\hat{P}|l, m_l\rangle = (-1)^l |l, m_l\rangle$, where \hat{P} is the parity operator.

a) [1] For each allowed value of s , what are the allowed values of l (up to $l=4$) consistent with the Spin-Statistics Theorem?

b) [3] The basis states above are not eigenstates of the total angular momentum, $\vec{J} = \vec{L} + \vec{S}$. We know how to construct states that are, using Clebsch-Gordan coefficients:

$$|n, l, s, j, m_j\rangle = \sum_{m_l+m_s=m_j} C_{m_l m_s m_j}^{l, s, j} |n, l, m_l\rangle \otimes |s, m_s\rangle$$

You don't need to look up any C-G coefficients for this problem. Instead, make a table with four columns showing all allowed combinations of s , l , and j , and the spectroscopic notation from atomic physics, $^{2S+1}L_J$.