

Introduction to Classical Mechanics

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1 HISTORY

Isaac Newton solved the premier scientific problem of his day, which was to explain the motion of the planets. He published his theory in the famous book known as *Principia*. The full Latin title of the book¹ may be translated into English as *Mathematical Principles of Natural Philosophy*.

The theory that the planets (including Earth) revolve around the sun was published by Nicolaus Copernicus in 1543. This was a revolutionary idea! The picture of the Universe that had been developed by astronomers before Copernicus had the Earth at rest at the center, and the sun, moon, planets and stars revolving around the Earth. But this picture failed to explain accurately the observed planetary positions. The failure of the Earth-centered theory led Copernicus to consider the sun as the center of planetary orbits. Later observations verified the Copernican theory. The important advances in astronomical observations were made by Galileo and Kepler.

Galileo Galilei was perhaps the most remarkable individual in the history of science. His experiments and ideas changed both physics and astronomy. In physics he showed that the ancient theories of Aristotle, which were still accepted in Galileo's time, are incorrect. In astronomy he verified the Copernican model of the Universe by making the first astronomical observations with a telescope.

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¹Philosophiae Naturalis Principia Mathematica

Galileo did not invent the telescope but he made some of the earliest telescopes, and his telescopes were the best in the world at that time. Therefore he discovered many things about the the solar system and stars:

- craters and mountains on the moon
- the moons of Jupiter
- the phases of Venus
- the motion of sunspots
- the existence of many faint stars

These discoveries provided overwhelming evidence in favor of the Copernican model of the solar system.

Johannes Kepler had extensive data on planetary positions, as functions of time, from observations collected earlier by Tycho Brahe. He analyzed the data based on the Copernican model, and deduced three empirical laws of planetary motion:

Kepler's Laws

1. The planets move on elliptical orbits with the sun at one focal point.
2. The radial vector sweeps out equal areas in equal times.
3. The square of the period of revolution is proportional to the cube of the semimajor axis of the ellipse.

Newton started with the results of Galileo and Kepler. His goal, then, was to explain why. *Why do the planets revolve around the sun in the manner discovered by Galileo and Kepler?* In particular, what is the explanation for the *mathematical regularities* in Kepler's laws of orbital motion? To answer this question, Newton had to develop the laws of motion and the theory of universal gravitation. And, to analyze the motion he invented a new branch of mathematics, which we now call *Calculus*.

The solution to planetary motion was published in *Principia* in 1687. Newton had solved the problem some years earlier, but kept it secret. He was visited in 1684 by the astronomer Edmund Halley. Halley asked what force would keep the planets in elliptical orbits. Newton replied that the force must be an inverse-square law, which he had proven by mathematical analysis; but he could not find the paper on which he had written the calculations! After further correspondence, Halley realized that Newton had made great advances in physics but had not published the results. With Halley's help, Newton published *Principia* in which he explained his theories of motion, gravity, and the solar system.

After the publication of *Principia*, Newton was the most renowned scientist in the world. His achievement was fully recognized during his lifetime. Today scientists and engineers still use Newton's theory of mechanics. In the 20th century some limitations of Newtonian mechanics were discovered: Classical mechanics breaks down for extreme speeds (approaching the speed of light) and at atomic dimensions. The theory of relativity, and quantum mechanics, were developed in the early 20th century to describe these cases. But for macroscopic systems Newton's theory is valid and extremely accurate.

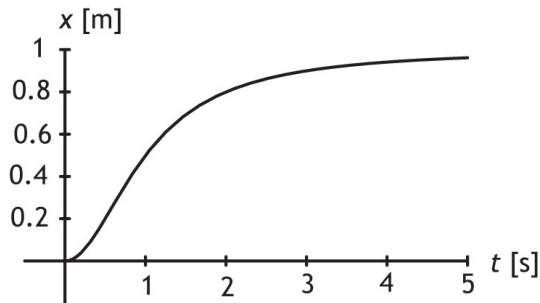
This early history of science is quite relevant to the study of calculus. Newton used calculus for analyzing motion, although he published the calculations in *Principia* using older methods of geometrical analysis. (He feared that the new mathematics—calculus—would not be understood or accepted.) Ever since that time, calculus has been necessary to the understanding of physics and its applications in science and engineering. So our study of mechanics will often require the use of calculus.

2 POSITION, VELOCITY, AND ACCELERATION

2.1 Position and velocity

Suppose an object M moves along a straight line.² We describe its motion by giving the position x as a function of time t , as illustrated in Fig. 1. The variable x is the *coordinate*, i.e., the displacement from a fixed point 0 called the *origin*. Physically, the line on which M moves might be pictured as a road, or a track. Mathematically, the positions form a representation of the ideal real line. The coordinate x is positive if M is to the right of 0 , or negative if to the left. The absolute value $|x|$ is the distance from 0 . The possible positions of M are in one-to-one correspondence with the set of real numbers. Hence *position* is a continuous function $x(t)$ of the independent variable t , time.

Figure 1: An example of position x as a function of time t for an object moving in one dimension. The object starts at rest at the origin at $t = 0$, begins moving to positive x , has positive acceleration for about 1 second, and then gradually slows to a stop at a distance of 1 m from the origin.



Example 2-1. What is the position as a function of time if M is at rest at a point 5 m to the left of the origin?

Solution. Because M is not moving, the function $x(t)$ is just a constant,

$$x(t) = -5 \text{ m.} \quad (1)$$

Note that the position has both a number (-5) and a unit (m, for meter). In this case the number is negative, indicating a position to the left of 0 . The number alone is not enough information. The unit is required. The unit may be changed by multiplying by a conversion factor. For example, the conversion from meters (m) to inches (in) is

$$5 \text{ m} = 5 \text{ m} \times \frac{38 \text{ in}}{1 \text{ m}} = 190 \text{ in.} \quad (2)$$

(There are 38 inches per meter.) So, the position could just as well be written as

$$x(t) = -190 \text{ in.} \quad (3)$$

Equations (3) and (1) are equivalent. This example shows why the number alone is not enough: The number depends on the unit.

²We'll call the moving object M . The letter M could stand for "moving" or "mass."

A Comment on Units of Measurement. In physical calculations it is important to keep track of the units of measurement—treating them as algebraic quantities. Dropping the unit will often lead to a failed calculation. Keeping the units has a bonus. It is a method of error checking. If the final unit is not correct, then there must be an error in the calculation; we can go back and figure out how to correct the calculation.

Example 2-2. A car travels on a straight road, toward the East, at a constant speed of 35 mph. Write the position as a function of time. Where is the car after 5 minutes?

Solution. The origin is not specified in the statement of the problem, so let's say that $x = 0$ at time $t = 0$. Then the position as a function of time is

$$x(t) = + \left(35 \frac{\text{mi}}{\text{hr}} \right) t, \quad (4)$$

where positive x is east of the origin. Equation (4) is based on the formula

$$\text{distance} = \text{speed} \times \text{time},$$

familiar from grade school; or, taking account of the signs,

$$\text{displacement} = \text{velocity} \times \text{time}.$$

Any distance must be positive. 'Displacement' may be positive or negative. Similarly, 'speed' must be positive, but 'velocity' may be positive or negative, negative meaning that M is moving to smaller x .

After 5 minutes, the position is

$$\begin{aligned} x(5 \text{ min}) &= 35 \frac{\text{mi}}{\text{hr}} \times 5 \text{ min} \\ &= 35 \frac{\text{mi}}{\text{hr}} \times 5 \text{ min} \times \frac{1 \text{ hr}}{60 \text{ min}} \\ &= 2.917 \text{ mi}. \end{aligned} \quad (5)$$

(We multiply by the conversion factor, 1 hr/60 min, to reduce the units.) The position could be expressed in feet, as

$$x(5 \text{ min}) = 2.917 \text{ mi} \times \frac{5280 \text{ ft}}{1 \text{ mi}} = 15400 \text{ ft}. \quad (6)$$

So, after 5 minutes the car is 15400 feet east of its initial position.

It is convenient to record some general, i.e., abstract formulas. If M is at rest at x_0 , then the function $x(t)$ is

$$x(t) = x_0. \quad (\text{object at rest}) \quad (7)$$

If M moves with constant velocity v_0 then

$$x(t) = x_0 + v_0 t. \quad (\text{object with constant velocity}) \quad (8)$$

Figure 2 illustrates graphs of these functions. The abscissa (horizontal axis) is the independent variable t , and the ordinate (vertical axis) is the dependent variable x . The slope in the second graph is v_0 . Note that the units of v_0 must be a length unit divided by a time unit, because

$$v_0 = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t}; \quad (9)$$

for example, the units could be m/s. Now, recall from calculus that the slope in a graph is equal to the derivative of the function! Thus, *the derivative of the position $x(t)$ is the velocity $v(t)$* .

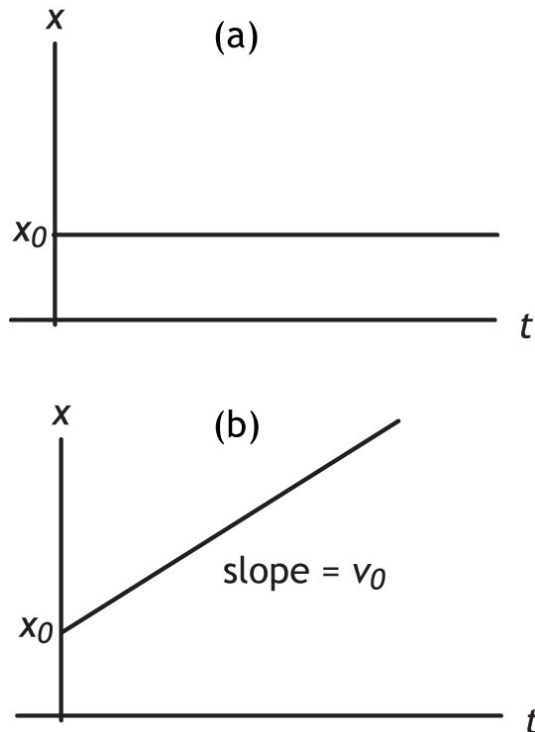


Figure 2: Motion of an object:
(a) zero velocity and (b) constant positive velocity.

So far we have considered only constant velocity. If the velocity is not constant, then the *instantaneous velocity* at a time t is the slope of the curve of x versus t , i.e., the slope of the tangent line. Again, this is precisely the derivative of $x(t)$. Letting $v(t)$ denote the velocity,

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \quad (10)$$

Another notation for the time-derivative, often used in mechanics, is $\dot{x}(t) \equiv dx/dt$.

Because $v(t)$ is defined by the limit $\Delta t \rightarrow 0$, there is an instantaneous velocity at every t . We summarize the analysis by a definition:

Definition. The velocity $v(t)$ is a function of time t , defined by
 $v(t) = dx/dt$.

2.2 Acceleration

If the velocity is changing then the object **M** is accelerating. The *acceleration* is defined as the time-derivative of the velocity,

$$a(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}. \quad (\text{definition of acceleration}) \quad (11)$$

By taking the limit $\Delta t \rightarrow 0$, the acceleration is defined at every instant. Also, because $v = dx/dt$, the acceleration a is the *second derivative* of $x(t)$,

$$a(t) = \frac{d^2x}{dt^2} = \ddot{x}(t). \quad (12)$$

Example 2-3. A car accelerates away from a stop sign, starting at rest. Assume the acceleration is a constant 5 m/s^2 for 3 seconds, and thereafter is 0. (a) What is the final velocity of the car? (b) How far does the car travel from the stop sign in 10 seconds?

Solution. (a) Let the origin be the stop sign. During the 3 seconds while the car is accelerating, the acceleration is constant and so the velocity function must be

$$v(t) = at, \quad (13)$$

because the derivative of at (with respect to t) is a . Note that $v(0) = 0$; i.e., the car starts from rest. At $t = 3 \text{ s}$, the velocity is the final velocity,

$$v_f = v(3 \text{ s}) = 5 \frac{\text{m}}{\text{s}^2} \times 3 \text{ s} = 15 \frac{\text{m}}{\text{s}}. \quad (14)$$

(b) The position as a function of time is $x(t)$, and

$$\frac{dx}{dt} = v(t). \quad (15)$$

Equation (15) is called a *differential equation* for $x(t)$. We know the derivative; what is the function? The general methods for solving differential equations involve integration. For this simple case the integral is elementary,

$$x(t) = x(0) + \int_0^t v(t) dt \quad (16)$$

Please be sure that you understand why (16) is correct.

general equations for constant acceleration	
position	$x(t) = x_0 + v_0t + \frac{1}{2}at^2$
velocity	$v(t) = dx/dt = v_0t + at$
acceleration	$a(t) = dv/dt = a, \text{ constant}$

Table 1: Formulae for constant acceleration. (x_0 = initial position, v_0 = initial velocity, and a = acceleration.)

Before we try to do the calculation, let's make sure we understand the problem. The car moves with velocity $v(t) = at$ (constant acceleration) for 3 seconds. How far does the car move during that time? Thereafter it moves with constant velocity $v_f = 15$ m/s. How far does it move from $t = 3$ s to $t = 10$ s? The combined distance is the distance traveled in 10 s.

Well, we can calculate the position by the integral in (16). The initial position is $x(0) = 0$. The velocity function is at for t from 0 to 3 s; and $v(t)$ is v_f for $t > 3$ s. Thus, for $t = 10$ s,

$$\begin{aligned} x(t) &= \int_0^3 at dt + \int_3^{10} v_f dt \\ &= \frac{1}{2}a(3 \text{ s})^2 + v_f(7 \text{ s}) \\ &= 127.5 \text{ m.} \end{aligned}$$

After 10 seconds the car has moved 127.5 meters.

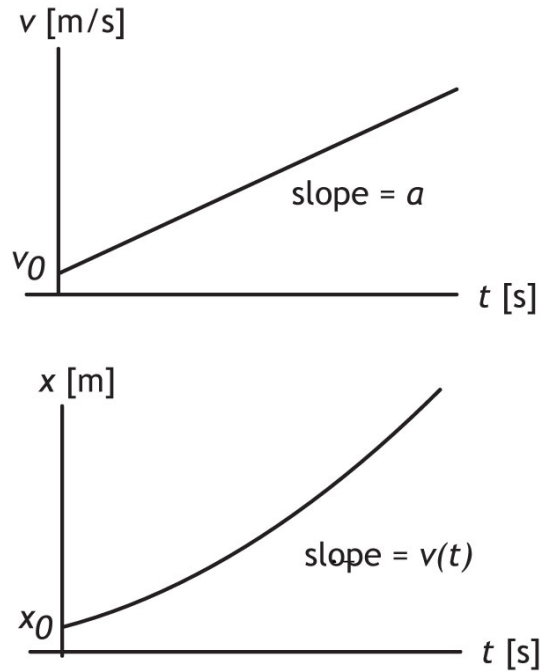
Generalization. Some useful general formulae for constant acceleration are recorded in Table 1. In the table, v_0 is a constant equal to the velocity at $t = 0$. Also, x_0 is a constant equal to the position at $t = 0$. As an exercise, please verify that $a = dv/dt$ and $v = dx/dt$. Remember that x_0 and v_0 are constants, so their derivatives are 0. The velocity and position as functions of t for constant acceleration are illustrated in Fig. 3.

Example 2-4. A stone is dropped from a diving platform 10 m high. When does it hit the water? How fast is it moving then?

Solution. We'll denote the height above the water surface by $y(t)$. The initial height is $y_0 = 10$ m. The initial velocity is v_0 ; because the stone is dropped, not thrown, its initial velocity v_0 is 0. The acceleration of an object in Earth's gravity, neglecting the effects of air resistance,³ is $a = -g$ where $g = 9.8$ m/s². The

³Air resistance is a frictional force called "drag," which depends on the size, shape, surface roughness, and speed of the moving object. The effect on a stone falling 10 m is small.

Figure 3: **Constant acceleration.** The two graphs are (a) velocity $v(t)$ and (b) position $x(t)$ as functions of time, for an object with constant acceleration a . Note that the slope increases with time in the lower graph. *Why?*



acceleration a is negative because the direction of acceleration is downward; i.e., the stone accelerates toward smaller y . Using Table 1, the equation for position y as a function of t is

$$y(t) = y_0 - \frac{1}{2}gt^2. \quad (17)$$

The variable y is the height above the water, so the surface is at $y = 0$. The time t_f when the stone hits the water is obtained by solving $y(t_f) = 0$,

$$y_0 - \frac{1}{2}gt_f^2 = 0. \quad (18)$$

The time is

$$t_f = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2 \times 10 \text{ m}}{9.8 \text{ m/s}^2}} = 1.43 \text{ s}. \quad (19)$$

Note how the final unit came out to be seconds, which is correct. The time to fall to the water surface is 1.43 seconds.

The equation for velocity is

$$v(t) = \frac{dy}{dt} = -gt. \quad (20)$$

This is consistent with the second row in Table 1, because $v_0 = 0$ and $a = -g$. The velocity when the stone hits the water is

$$v_f = v(t_f) = -9.8 \frac{\text{m}}{\text{s}^2} \times 1.43 \text{ s} = -14.0 \frac{\text{m}}{\text{s}}. \quad (21)$$

The velocity is negative because the stone is moving downward. The final *speed* is the absolute value of the velocity, 14.0 m/s.

2.3 Newton's second law

Newton's second law of motion states that the acceleration \mathbf{a} of an object is proportional to the *net force* \mathbf{F} acting on the object,

$$\mathbf{a} = \frac{\mathbf{F}}{m}, \quad (22)$$

or $\mathbf{F} = m\mathbf{a}$. The constant of proportionality m is the *mass* of the object. Equation (22) may be taken as the *definition* of the quantity m , the mass.

A Comment on Vectors. For two- or three-dimensional motion, the position, velocity, and acceleration are all vectors—mathematical quantities with both magnitude and direction. We will denote vectors by **boldface** symbols, e.g., \mathbf{x} for position, \mathbf{v} for velocity, and \mathbf{a} for acceleration. In hand-written equations, vector quantities are usually indicated by drawing an arrow (\rightarrow) over the symbol.

Acceleration is a kinematic quantity—determined by the motion. Equation (22) relates acceleration and force. *But some other theory must determine the force.* There are only a few basic forces in nature: gravitational, electric and magnetic, and nuclear. All observed forces (e.g., contact, friction, a spring, atomic forces, etc.) are produced in some way by those basic forces. Whatever force is acting on an object, (22) states how that force influences the motion of the object (according to classical mechanics!).

The mass m in (22) is called the *inertial mass*, because it would be determined by measuring the acceleration produced by a given force. For example, if an object is pulled by a spring force of 50 N, and the resulting acceleration is measured to be 5 m/s^2 , then the mass is equal to 10 kg.

The gravitational force is exactly (i.e., as precisely as we can measure it!) proportional to the inertial mass. Therefore the acceleration due to gravity is independent of the mass of the accelerating object. For example, at the surface of the Earth, all falling objects have the same acceleration due to gravity, $g = 9.8 \text{ m/s}^2$ (ignoring the force of air resistance⁴). It took the great genius of Galileo to see that the small differences between falling objects are not an effect of gravity but of air resistance. Newton's equation $\mathbf{a} = \mathbf{F}/m$ explains why: The *force* of gravity is proportional to the mass; therefore the *acceleration* by gravity is independent of the mass.

The equation $\mathbf{F} = m\mathbf{a}$ tells us how an object will respond to a specified force. Because the acceleration \mathbf{a} is a derivative,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2}, \quad (23)$$

⁴Take a sheet of paper and drop it. It falls slowly and irregularly, not moving straight down but fluttering this way and that, because of aerodynamic forces. But wad the same piece of paper up into a small ball and drop it. Then it falls with the same acceleration as a more massive stone.

Newton's second law is a *differential equation*. To use Newtonian mechanics, we must solve differential equations.

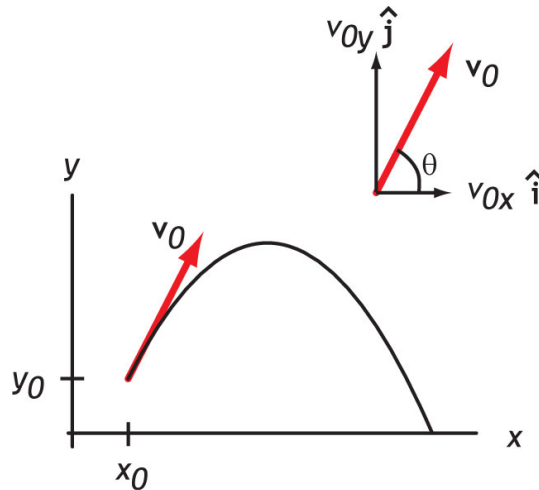
3 PROJECTILE MOTION

An ideal projectile is an object M moving in Earth's gravity with no internal propulsion, and no external forces except gravity. (A *real* projectile is also subject to aerodynamic forces such as drag and lift. We will neglect these forces, a fairly good approximation if M moves slowly.)

The motion of a projectile must be described with *two* coordinates: horizontal (x) and vertical (y). Figure 4 shows the motion of the projectile in the xy coordinate system. The curve is the *trajectory* of M .

Figure 4: **Projectile motion.**

Horizontal (x) and vertical (y) axes are set up to analyze the motion. The initial position is $(x, y) = (x_0, y_0)$. The initial velocity \mathbf{v}_0 is shown as a vector at (x_0, y_0) . The curve is the trajectory of the projectile. The inset shows the initial velocity vector \mathbf{v}_0 separated into horizontal and vertical components, $v_{0x}\hat{\mathbf{i}}$ and $v_{0y}\hat{\mathbf{j}}$; θ is the angle of elevation of \mathbf{v}_0 . ($\hat{\mathbf{i}}$ = unit horizontal vector, $\hat{\mathbf{j}}$ = unit vertical vector)



Suppose M is released at $(x, y) = (x_0, y_0)$ at time $t = 0$. Figure 4 also indicates the initial velocity vector \mathbf{v}_0 which is tangent to the trajectory at (x_0, y_0) . Let θ be the angle of elevation of the initial velocity; then the x and y components of the initial velocity vector are

$$v_{0x} = v_0 \cos \theta, \quad (1)$$

$$v_{0y} = v_0 \sin \theta. \quad (2)$$

Horizontal component of motion. The equations for the horizontal motion are

$$x(t) = x_0 + v_{0x}t, \quad (3)$$

$$v_x(t) = v_{0x}. \quad (4)$$

These are the equations for constant velocity, $v_x = v_{0x}$. There is no horizontal acceleration (neglecting air resistance) because the gravitational force is vertical.

Vertical component of motion. The equations for the vertical motion are

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2, \quad (5)$$

$$v_y(t) = v_{0y} - gt. \quad (6)$$

These are the equations for constant acceleration, $a_y = -g$. (As usual, $g = 9.8 \text{ m/s}^2$.) The vertical force is $F_y = -mg$, negative indicating downward, where m is the mass of the projectile. The acceleration is $a_y = F_y/m$ by Newton's second law.⁵ Thus $a_y = -g$. The acceleration due to gravity does not depend on the mass of the projectile because the force is proportional to the mass. Thus (5) and (6) are independent of the mass.

Example 3-5. Verify that the derivative of the position vector is the velocity vector, and the derivative of the velocity vector is the acceleration vector, for the projectile.

Solution. Position, velocity, and acceleration are all *vectors*. The position vector is

$$\mathbf{x}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}. \quad (7)$$

Here $\hat{\mathbf{i}}$ denotes the horizontal unit vector, and $\hat{\mathbf{j}}$ denotes the vertical unit vector. For the purposes of describing the motion, these unit vectors are *constants*, independent of t . The time dependence of the position vector $\mathbf{x}(t)$ is contained in the coordinates, $x(t)$ and $y(t)$.

The derivative of $\mathbf{x}(t)$ is

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} \\ &= v_{0x}\hat{\mathbf{i}} + (v_{0y} - gt)\hat{\mathbf{j}} \\ &= v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} = \mathbf{v}. \end{aligned} \quad (8)$$

As required, $d\mathbf{x}/dt$ is \mathbf{v} . The derivative of $\mathbf{v}(t)$ is

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} \\ &= 0 + (-g)\hat{\mathbf{j}} = -g\hat{\mathbf{j}}. \end{aligned} \quad (9)$$

The acceleration vector has magnitude g and direction $-\hat{\mathbf{j}}$, i.e., downward; so $\mathbf{a} = -g\hat{\mathbf{j}}$. We see that $d\mathbf{v}/dt = \mathbf{a}$, as required.

3.1 Summary

To describe projectile motion (or 3D motion in general) we must use vectors. However, for the ideal projectile (without air resistance) the two components—horizontal and vertical—are independent. The horizontal component of the motion

⁵Newton's second law is $\mathbf{F} = m\mathbf{a}$.

has constant velocity v_{0x} , leading to Eqs. (3) and (4). The vertical component of the motion has constant acceleration $a_y = -g$, leading to Eqs. (5) and (6).

To depict the motion, we could plot $x(t)$ and $y(t)$ versus t separately, or make a parametric plot of y versus x with t as independent parameter.⁶ The parametric plot yields a *parabola*. Galileo was the first person to understand the trajectory of an ideal projectile (with negligible air resistance): The trajectory is a parabola.

⁶This kind of plot — $y(t)$ versus $x(t)$ with t as independent parameter — is called a parametric plot.

4 CIRCULAR MOTION

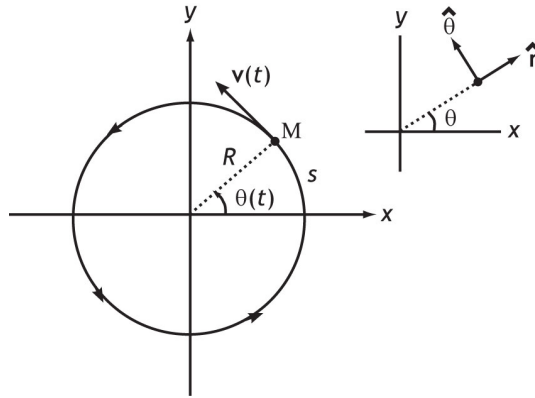
Consider an object M moving on a circle of radius R , as illustrated in Fig. 5. We could describe the motion by Cartesian coordinates, $x(t)$ and $y(t)$, but it is simpler to use the angular position $\theta(t)$ because the radius R is constant. The angle θ is defined in Fig. 5. It is the angle between the radial vector and the x axis. The value of θ is sufficient to locate M . From Fig. 5 we see that the Cartesian coordinates are

$$x(t) = R \cos \theta(t), \quad (1)$$

$$y(t) = R \sin \theta(t). \quad (2)$$

If $\theta(t)$ is known, then $x(t)$ and $y(t)$ can be calculated from these equations.

Figure 5: **Circular motion.** A mass M moves on a circle of radius R . The angle $\theta(t)$ is used to specify the position. In radians, $\theta = s/R$ where s is the arclength, as shown. The velocity vector $\mathbf{v}(t)$ is tangent to the circle. The inset shows the unit vectors $\hat{\theta}$ and \hat{r} , which point in the direction of increasing θ and r , respectively.



In calculus we always use the *radian measure* for an angle θ . The radian measure is defined as follows. Consider a circular arc with arclength s on a circle of radius R . The angle subtended by the arc, in radians, is

$$\theta = \frac{s}{R}. \quad (\text{radian measure}) \quad (3)$$

4.1 Angular velocity and the velocity vector

The angular velocity $\omega(t)$ is defined by

$$\omega(t) = \frac{d\theta}{dt}. \quad (\text{angular velocity}) \quad (4)$$

This function is the *instantaneous* angular velocity at time t . For example, if M moves with constant speed, traveling around the circle in time T , then the angular velocity is constant and given by

$$\omega = \frac{2\pi}{T}. \quad (\text{constant angular velocity}) \quad (5)$$

To derive (5) consider the motion during a time interval Δt . The arclength Δs traveled along the circle during Δt is $R\Delta\theta$ where $\Delta\theta$ is the change of θ during Δt , in radians. The angular velocity is then

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\Delta s/R}{\Delta t}. \quad (6)$$

Because the speed is constant, $\Delta\theta/\Delta t$ is constant and independent of the time interval Δt . Let Δt be one period of revolution T . The arclength for a full revolution is the circumference $2\pi R$. Thus

$$\omega = \frac{2\pi R/R}{T} = \frac{2\pi}{T}. \quad (7)$$

The instantaneous *speed* of the object is the rate of increase of distance with time,

$$\begin{aligned} v(t) &= \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{R\Delta\theta}{\Delta t} \\ &= R \frac{d\theta}{dt} = R\omega(t). \end{aligned} \quad (8)$$

But what is the instantaneous *velocity*? Velocity is a vector \mathbf{v} , with both direction and magnitude. The magnitude of \mathbf{v} is the speed, $v = R\omega$. The direction is tangent to the circle, which is the same as the unit vector $\hat{\boldsymbol{\theta}}$. (See Fig. 5.) Thus the velocity vector is

$$\mathbf{v} = R\omega\hat{\boldsymbol{\theta}}, \quad (9)$$

which points in the direction of $\hat{\boldsymbol{\theta}}$ and has magnitude $R\omega$. In general, \mathbf{v} , ω , θ and $\hat{\boldsymbol{\theta}}$ are all functions of time t as the particle moves around the circle. But of course for circular motion, R is constant. We summarize our analysis as a theorem:

Theorem 1. The velocity vector in circular motion is

$$\mathbf{v}(t) = R\omega(t)\hat{\boldsymbol{\theta}}(t). \quad (10)$$

4.2 Acceleration in circular motion

Now, what is the *acceleration* of M as it moves on the circle? The acceleration \mathbf{a} is a vector, so we must determine both its magnitude and direction. Unlike the velocity \mathbf{v} , which must be tangent to the circle, the acceleration has both tangential and radial components.

Recall that we have defined acceleration as the derivative of velocity in the case of one-dimensional motion. The same definition applies to the vector quantities for two- or three-dimensional motion. Using the definition of the derivative,

$$\mathbf{a}(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d\mathbf{v}}{dt}. \quad (11)$$

The next theorem relates \mathbf{a} for circular motion to the parameters of the motion.

Theorem 2. The acceleration vector in circular motion is

$$\mathbf{a} = R \frac{d\omega}{dt} \hat{\boldsymbol{\theta}} - R\omega^2 \hat{\mathbf{r}}. \quad (12)$$

Proof: We must calculate the derivative of \mathbf{v} , using (10) for \mathbf{v} . At time t , the acceleration is

$$\begin{aligned} \mathbf{a}(t) &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt} (R\omega\hat{\boldsymbol{\theta}}) \\ &= R \left(\frac{d\omega}{dt} \hat{\boldsymbol{\theta}} + \omega \frac{d\hat{\boldsymbol{\theta}}}{dt} \right). \end{aligned} \quad (13)$$

Note that (13) follows from the Leibniz rule for the derivative of the product $\omega(t)\hat{\boldsymbol{\theta}}(t)$. Now,

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\hat{\boldsymbol{\theta}}}{\Delta t}. \quad (14)$$

Figure 6 demonstrates that $\Delta\hat{\boldsymbol{\theta}} \approx -\hat{\mathbf{r}}\Delta\theta$ for small $\Delta\theta$. (The relation of differentials is $d\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}d\theta$.) The direction of $\Delta\hat{\boldsymbol{\theta}}$ is *radially inward*. This little result has interesting consequences, as we'll see! The derivative is then

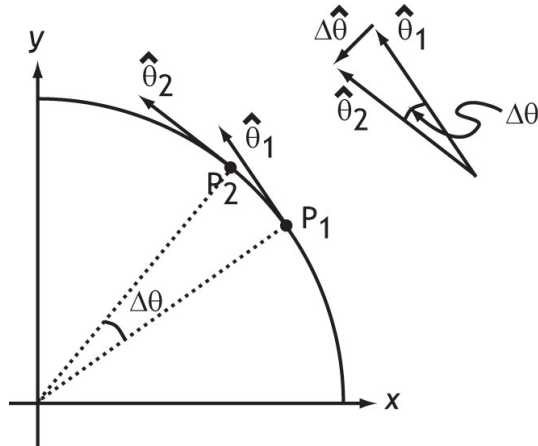
$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{-\hat{\mathbf{r}}\Delta\theta}{\Delta t} = -\hat{\mathbf{r}} \frac{d\theta}{dt} = -\hat{\mathbf{r}}\omega. \quad (15)$$

Substituting this result into (13) we find

$$\mathbf{a}(t) = R \frac{d\omega}{dt} \hat{\boldsymbol{\theta}} - R\omega^2 \hat{\mathbf{r}}, \quad (16)$$

which proves the theorem.

Figure 6: Proof that $d\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}d\theta$. P_1 and P_2 are points on the circle with angle difference $\Delta\theta$. The inset shows that $\Delta\hat{\boldsymbol{\theta}} (= \hat{\boldsymbol{\theta}}_2 - \hat{\boldsymbol{\theta}}_1)$ is centripetal (i.e., in the direction of $-\hat{\mathbf{r}}$) and has magnitude $\Delta\theta$ in the limit of small $\Delta\theta$.



For circular motion, the radial component of the acceleration vector is $a_r = -R\omega^2$. This component of \mathbf{a} is called the *centripetal acceleration*. The word “centripetal” means *directed toward the center*. We may write a_r in another form. By Theorem 1, $\omega = v/R$; therefore

$$a_r = -\frac{v^2}{R}. \quad (17)$$

If the speed of the object is constant, then $d\omega/dt = 0$ and the acceleration \mathbf{a} is purely centripetal. In *uniform circular motion*, the acceleration vector is always directed toward the center of the circle with magnitude v^2/R .

Imagine a ball attached to a string of length R , moving around a circle at constant speed with the end of the string fixed. The trajectory must be a circle because the string length (the distance from the fixed point) is constant. *The ball constantly accelerates toward the center of the circle ($a_r = -v^2/R$) but it never gets any closer to the center ($r(t) = R$, constant)! This example illustrates the fact that the velocity and acceleration vectors may point in different directions. In uniform circular motion, the velocity is tangent to the circle but the acceleration is centripetal, i.e., orthogonal to the velocity.*

Example 4-6. Suppose a race car travels on a circular track of radius $R = 50$ m. (This is quite small!) At what speed is the centripetal acceleration equal to $1g$?

Solution. Using the formula $a = v^2/R$, and setting $a = g$, the speed is

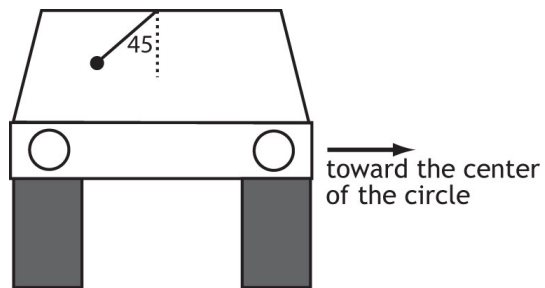
$$v = \sqrt{gR} = \sqrt{9.8 \text{ m/s}^2 \times 50 \text{ m}} = 22.1 \text{ m/s}. \quad (18)$$

Converting to miles per hour, the speed is about 48 mi/hr. A pendulum suspended from the ceiling of the car would hang at an angle of 45 degrees to the vertical (in equilibrium), because the horizontal and vertical components of force exerted by the string on the bob would be equal, both equal to mg . The pendulum would hang *outward* from the center of the circle, as shown in Fig. 7. Then the string exerts a force on the bob with an *inward* horizontal component, which is the centripetal force on the bob.

★

The equation $a_r = -v^2/r$ for the centripetal acceleration in circular motion was first published by Christiaan Huygens in 1673 in a book entitled *Horologium Oscillatorium*. Huygens, a contemporary of Isaac Newton, was one of the great figures of the Scientific Revolution. He invented the earliest practical pendulum clocks (the main subject of the book mentioned). He constructed excellent telescopes, and discovered that the planet Saturn is encircled by rings. In his scientific work, Huygens was guided by great skill in mathematical analysis. Like Galileo and Newton, Huygens used mathematics to describe nature accurately.

Figure 7: A race car on a circular track has centripetal acceleration v^2/R . If $v^2/R = g$, then the equilibrium of a pendulum suspended from the ceiling is at 45 degrees to the vertical. In the frame of reference of the track, the bob accelerates centripetally because it is pulled toward the center by the pendulum string. In the frame of reference of the car there is a centrifugal force—an apparent (but fictitious) force directed away from the center of the track.

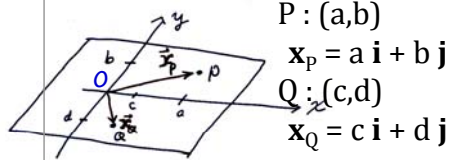


Introduction and Review Vectors Review

A vector is a mathematical quantity with both a **direction** and a **magnitude**.

Example

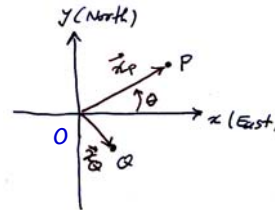
Position vectors in 2 dimens.



Distance O to P = $\sqrt{a^2 + b^2}$

Distance Q to P =
 $\sqrt{(a-c)^2 + (b-d)^2}$

Directions



Vector \mathbf{x}_P is at angle θ North of East;
 $\theta = \arctan(b/a)$.

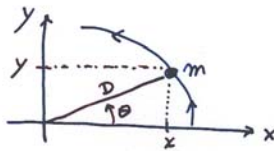
The position of a particle is given by the position vector $\mathbf{x}_P = a \mathbf{i} + b \mathbf{j}$. The coordinates (a, b) are the displacements from the origin, O.

1d - Vectors Review

1

Vectors and Scalars

Position Vectors



$\mathbf{r} = x \hat{\mathbf{i}} + y \hat{\mathbf{j}}$ position vector

$D = \sqrt{x^2 + y^2}$ is the magnitude

$\theta = \arctan(y/x)$ is the direction

Velocity is a vector,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt} \hat{\mathbf{i}} + \frac{dy}{dt} \hat{\mathbf{j}}$$

Time is a scalar, t.

Vectors

$\mathbf{r}, \mathbf{v}, \mathbf{a}$ (kinematics)

$\mathbf{F}, \mathbf{p}, \mathbf{L}$ (dynamics)

Scalars

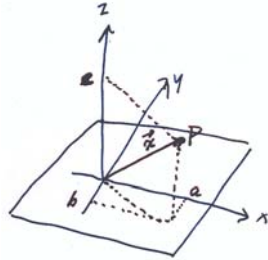
- distance $r = |\mathbf{r}|$
- speed $v = |\mathbf{v}|$
- magnitude of a vector $A = |\mathbf{A}|$
- time t
- mass m
- energy E

A scalar is a mathematical quantity that has no direction (in space).

1d - Vectors Review

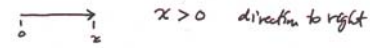
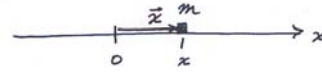
2

1, 2, 3 dimensions

Vectors in 3 dimensions

$P: (a, b, c)$ Cartesian coordinates

$$\mathbf{x}_p = a \mathbf{i} + b \mathbf{j} + c \mathbf{k}$$

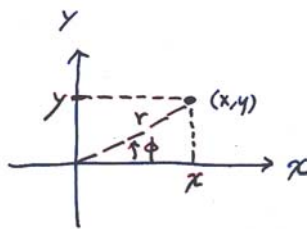
Vectors in 1 dimension

In one dimension, there are only two directions (+x or -x)

1d - Vectors Review

3

Coordinate Systems

Plane polar coordinates (2D)

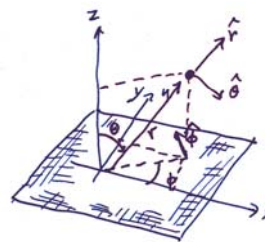
$$r = \sqrt{x^2 + y^2}$$

$$\phi = \arctan(y/x)$$

$$\mathbf{x} = x \hat{\mathbf{i}} + y \hat{\mathbf{j}} = r \hat{\mathbf{r}}$$

E.g., circular motion

$$\begin{aligned} x &= r \cos \phi \\ y &= r \sin \phi \end{aligned}$$

Spherical Polar Coordinates (3D)

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = \arctan(y/x)$$

$$\theta = \arctan(r_{\perp}/r)$$

$$\mathbf{x} = x \hat{\mathbf{i}} + y \hat{\mathbf{j}} + z \hat{\mathbf{k}} = r \hat{\mathbf{r}}$$

E.g., motion with a central force

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned}$$

1d - Vectors Review

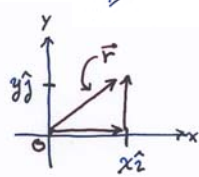
4

Vector Algebra

Addition of Vectors

$$\mathbf{A} + \mathbf{B}$$

$$\text{e.g., } \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$



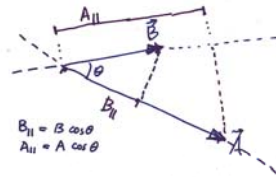
Multiplication of Vectors

Dot Product $\mathbf{A} \cdot \mathbf{B}$ -- a scalar
 Cross Product $\mathbf{A} \times \mathbf{B}$ -- a vector

Dot Product

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta$$

$\mathbf{A} \cdot \mathbf{B}$ involves the *projection* of one vector on the other.



In Cartesian coordinates,
 $\mathbf{A} \cdot \mathbf{B} = (a_x \hat{\mathbf{i}} + a_y \hat{\mathbf{j}}) \cdot (b_x \hat{\mathbf{i}} + b_y \hat{\mathbf{j}})$
 $= a_x b_x + a_y b_y$

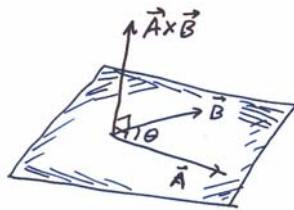
1d - Vectors Review

5

Vector Algebra

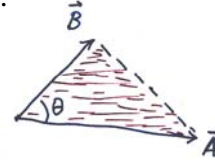
Cross Product

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta \hat{\mathbf{n}}$$

 $\mathbf{A} \times \mathbf{B}$

- Direction is \perp to the plane;
- Magnitude = $AB |\sin \theta|$.

$\frac{1}{2} |\mathbf{A} \times \mathbf{B}|$ = the area inside the triangle:

Proof

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times A \times B \sin \theta \\ &= \frac{1}{2} |\mathbf{A} \times \mathbf{B}| \end{aligned}$$

In Cartesian coordinates,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

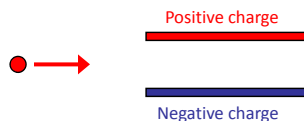
1d - Vectors Review

6

Calculus is the branch of mathematics that describes continuous change. It is an essential part of physics, because most laws of physics involve continuous change.

1

Calculate the motion of the particle:



2

We'll review calculus quickly, for its applications in mechanics. Students enrolled in PHY 321 should have taken at least two semesters of calculus, including both differentiation and integration.

3

Calculus was first developed by Isaac Newton, and independently by Gottfried Leibniz. Newton needed this new mathematics for his laws of motion.

4

For example, Newton's second law is a differential equation (or actually, a pair of equations ...

$$\frac{dp}{dt} = F \quad \text{and} \quad p = mv = m \frac{dx}{dt}$$

5

To apply these equations, and solve them, we'll need a good understanding of calculus. For example,

$$\Delta p = \int F(t) dt \quad ; \quad \text{or,} \quad \Delta x = \int v(t) dt$$

6

Derivative:
Slope and Rate of Change

1

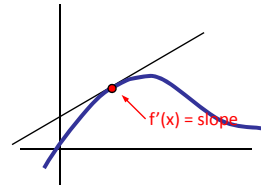
The definition of the derivative of a function $f(x)$ is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Another notation for $f'(x)$ is $\frac{df}{dx}$

2

A useful geometric picture of the derivative is the slope of a graph of $f(x)$. That is, $f'(x)$ is the slope of the line tangent to the graph of $f(x)$ at x .

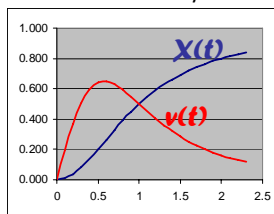


3

Rate of change. A function $f(x)$ has an independent variable x and a dependent variable f . The derivative $f'(x)$ is the rate of change of f with respect to x . For example, in mechanics we write $x(t)$ for the position as a function of time; the velocity = rate of change of position; $v(t) = dx/dt$

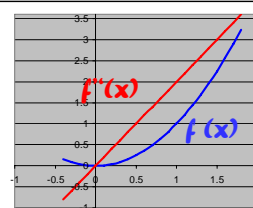
4

Position and velocity

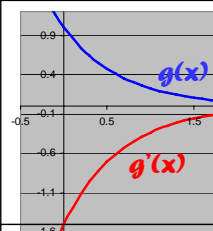
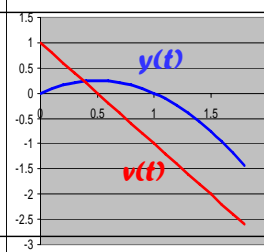


5

Example 1. $f(x) = x^2$. Calculate $f'(x)$.

**1**

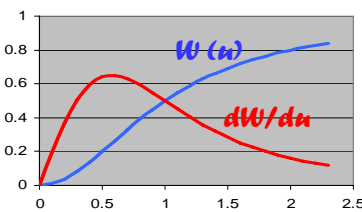
Example 2. $f(x) = e^{-1.5x}$. Calculate $g'(x)$.

**2**

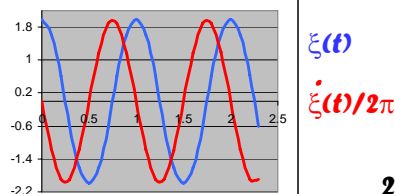
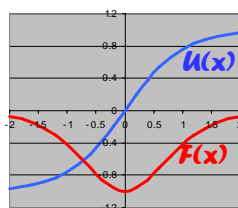
Example 3. $y(t) = v_0 t - 0.5 g t^2$.
Calculate $v_y t$.

3

Example 4. $W(u) = u^2$.
Calculate dW/du .

**1**

Example 5. $\xi(t) = 2 \cos(2\pi t)$.
Calculate $\dot{\xi}(t)$.

**2**

Example 6.
The potential energy is
 $U(x) = \tanh(x)$.
Calculate the force $F(x)$.

3

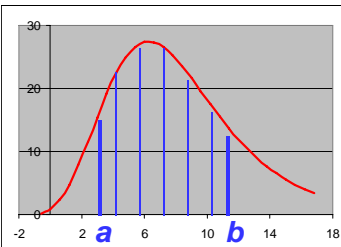
Integral:**Area or Total of a Continuum**

The definition of the integral of a function $g(x)$, over a domain $a < x < b$, is

$$\int_a^b g(x) dx = \lim_{N \rightarrow \infty} \sum_{n=1}^N g(x_n) \delta x$$

where $\delta x = (b-a)/N$ and $x_n = a + (n-1/2)\delta x$

1



The integral as the "area under the curve".

2

Integration is the process of calculation in which a large number of small components are added together, in the limit that the number of parts approaches infinity and the sizes approach 0.

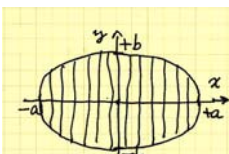
3

We often encounter integrals in physics, because we must calculate the total of some continuum quantity. We subdivide the continuum into infinitesimal parts, and add their contributions –

integration.

4

Example. Calculate the area of an ellipse with semi-major axis = a and semi-minor axis = b .



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

5

$$\begin{aligned} \text{Area} &= \int_{-a}^a 2y dx \\ &\Rightarrow \int_{-a}^a 2b \sqrt{1 - x^2/a^2} dx \\ &= 2b \int_{-\pi/2}^{\pi/2} \cos \theta \cdot a \cos \theta d\theta \quad \left\{ \begin{array}{l} x = a \sin \theta \\ dx = a \cos \theta d\theta \end{array} \right. \\ &= 2ab \left\{ \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right\} \Big|_{-\pi/2}^{\pi/2} \\ &= 2ab \cdot \frac{\pi}{2} \quad \boxed{\text{Area} = \pi ab} \end{aligned}$$