Oscillations

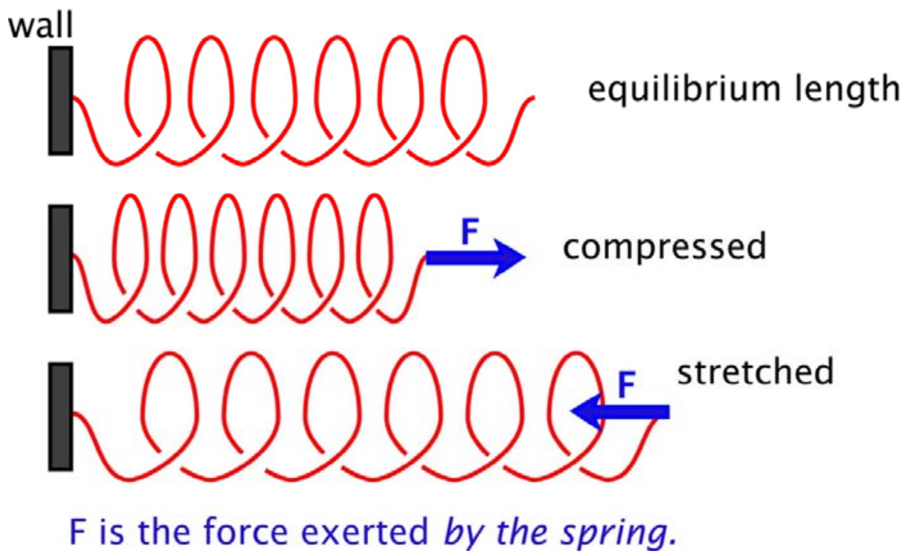
4a. The Simple Harmonic Oscillator

In general, an oscillating system with sinusoidal time dependence is called a harmonic oscillator. Many physical systems have this time dependence: mechanical oscillators, elastic systems, AC electric circuits, sound vibrations, etc.

<u>Spring Forces</u>

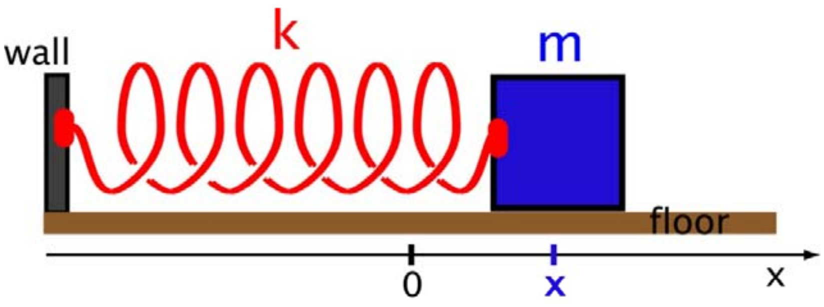
Robert Hooke, a contemporary of Isaac Newton (*), found that spring forces can be described by some simple properties ...

- The spring has an equilibrium length.
- If stretched or compressed by a small displacement, x, a restoring force pulls or pushes the spring toward equilibrium length.
- Within the elastic limit, the force is linear in the displacement; $F(x) = -k\,x$.



(*) Hooke and Newton were acquaintances but their relationship was not friendly.

Dynamics of a mass on a spring



The motion of the block is an example of single-particle dynamics and one dimensional motion.

The equation of motion is $m\ddot{x} = F(x) = -kx$

Or, $\ddot{x} = -\omega^2 x$ where $\omega = \sqrt{\frac{k}{m}}$

Example 1. General initial conditions.

Suppose we are given initial conditions,

$$x(0) = x_0 \quad \text{and} \quad v(0) = v_0.$$

Example 2. Suppose the initial position is x = 0, and the amplitude of oscillator is R. Then what is x(t)?

Example 3. The mass is 1 kg and Hooke's constant is 100 N/cm. What is the frequency of oscillation?

Example 4. Energy

Consider the general solution

$$x(t) = A \cos \omega t + B \sin \omega t$$

(A) Calculate the kinetic and potential energy.

(B) Calculate the time averages of K and U.

(C) Calculate E and verify that it is constant.

(D) Calculate the amplitude of oscillation.

$$\boxed{x(t) = A\cos\omega t + B\sin\omega t}$$

(A) Kinetic and Potential Energy are...

$$K = \frac{1}{2}m\dot{x}^2 = \frac{m}{2}\left(-\omega A\sin\omega t + \omega B\cos\omega t\right)^2$$

$$= \frac{1}{2}m\omega^2\left(A^2\sin^2\omega t + B^2\cos^2\omega t - 2AB\sin\omega t\cos\omega t\right)$$

$$U = \frac{1}{2}kx^2 = \frac{k}{2}\left(A\cos\omega t + B\sin\omega t\right)^2$$

$$= \frac{1}{2}m\omega^2\left(A^2\cos^2\omega t + B^2\sin^2\omega t + 2AB\sin\omega t\cos\omega t\right)$$

(note : $k = m\omega^2$)

(B) Time averages of K and U are...

$\langle\sin^2\omega t\rangle = \langle\cos^2\omega t\rangle = \frac{1}{2}$

$\langle\sin\omega t\cos\omega t\rangle = 0$

$$\langle K\rangle = \frac{1}{2}m\omega^2\left(\frac{1}{2}A^2 + \frac{1}{2}B^2\right)$$

$$\langle U\rangle = \frac{1}{2}m\omega^2\left(\frac{1}{2}A^2 + \frac{1}{2}B^2\right)$$

(C) The total energy is $E = K + U$

$$E = \frac{1}{2}m\omega^2\left(A^2 + B^2\right)$$

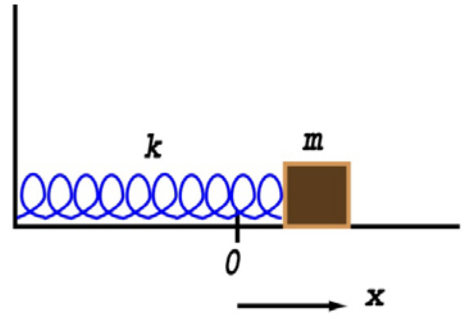
and note that E is constant in time.

(D) Let $\underline{\underline{R}}$ denote the amplitude of oscillation. When the mass is at its maximum displacement, i.e., $x = R$, the velocity is 0. So at $x = R$, the kinetic energy is 0 and the potential energy is $U = E$.

At maximum displacement: $U = \frac{1}{2}kR^2 = E$ $= \frac{1}{2}m\omega^2(A^2 + B^2)$. [Remember : $k = m\omega^2$.] Thus

$$R = \sqrt{A^2 + B^2}\,.$$

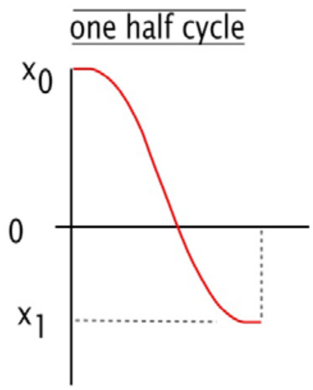
Example: Sliding Friction
Now suppose the coefficient of (kinetic) friction for the block sliding on the surface is μ. Determine x(t).



1

First consider one half cycle of the oscillation. The coordinate x(t) varies from positive amplitude x_0 to negative amplitude x_1. The velocity is instantaneously 0 at x_0 and x_1.

2



3

During this part of the motion,

$$m\ddot{x} = -kx + \mu mg$$

(The block is moving to the left so the frictional force is toward the right, +μmg.)

4

The solution is

$$x(t) = \frac{\mu mg}{k} + A\cos\omega t + B\sin\omega t$$

where the initial conditions require

$$A = \left(x_0 - \frac{\mu mg}{k}\right) \quad \text{and} \quad B = 0$$

5

The negative amplitude, which occurs at $\omega t = \pi$, is

$$x_1 = \frac{\mu mg}{k} - A = -x_0 + \frac{2\mu mg}{k}$$

Thus ***the amplitude of oscillation decreases by 2μmg/k for each half cycle.***

Exercise: Analyze the energy. Calculate the change of potential energy for one half cycle (from max displacement x_0 to min displacement x_1) and the work done by friction.
Show that $\Delta U = W$.

Epilogue (4a)

Epilog (4a) - Complex Exponential Functions

$$x = -\omega^2 x \qquad (1)$$

The general solution of the harmonic oscillator equation (1) may be written in several ways. In the lecture I wrote

$$x(t) = A \cos \omega t + B \sin \omega t \qquad (2)$$

which has two parameters (A, B) which can be adjusted to match initial values or other information about the motion.

Another form of the general solution is

$$x(t) = C \cos (\omega t - \phi) \qquad (3)$$

which also has two adjustable parameters (C = amplitude and ϕ = phase shift).

To see that either (2) or (3) can be used as a general solution of (1), note that (3) could be written

$$x(t) = C \cos \phi \ \cos \omega t \ + \ C \sin \phi \ \sin \omega t \qquad (4)$$

which has the same form as (2), with

$$A = C \cos \phi \quad \text{and} \quad B = C \sin \phi. \qquad (5)$$

Sinusoidal Functions and Complex Exponentials

$$\ddot{x} = -\omega^2 x \qquad (1)$$

We could also write a general solution of (1) as a linear combination of complex exponentials,

$$x(t) = \alpha\, e^{\iota\omega t} + \beta\, e^{-\iota\omega t} \;, \qquad (6)$$

where α and β are complex numbers. [$\iota = \text{sqrt}(-1)$]
(Recall Euler's equation.)

But x must be <u>real</u>. It's the displacement of the mass from equilibrium, which can't be a complex number. So why would we introduce complex numbers into the solution, if we know the solution must be real?

The reason for using complex exponentials (which is common in physics) is that calculations may be simpler with exponentials (even complex exponentials than with sines and cosines. So we write the solution using complex functions and parameters for intermediate calculations. But at the end of hth calculations, we must take the real part of the expressions to get the physical solutions. The trick is : ***take the real part at the end***.

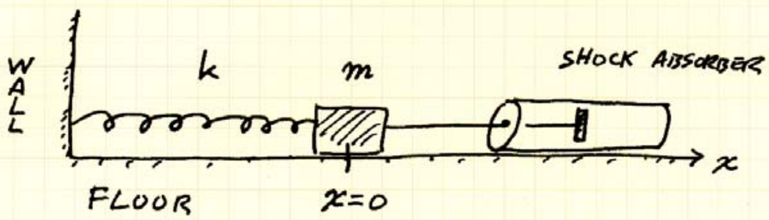
<u>Euler's equation and related equations</u>

$$e^{\iota\theta} = \cos\theta + i\sin\theta \qquad e^{-\iota\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{\iota\theta} + e^{-\iota\theta}}{2} \qquad \sin\theta = \frac{e^{\iota\theta} - e^{-\iota\theta}}{2\iota}$$

Oscillations

4b – The Damped Oscillator



$$m \frac{dv}{dt} = -kx - bv$$

Restoring force; Hooke's law

Resistance force; viscosity

Standardized form

$$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = 0$$

where $2\beta = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$

Special limiting cases

(1) $k = 0 \Rightarrow \frac{dv}{dt} = -2\beta v$

$$v(t) = C\, e^{-2\beta t}$$ exponential damping

(2) $b = 0 \Longrightarrow \ddot{x} = -\omega_0^2 x$

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

simple harmonic motion

General Solution, using exponential functions

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \ .$$

Try $x(t) = Ce^{rt}$ (Appendix C)

$$Cr^2e^{rt} + 2\beta Cre^{rt} + \omega_0^2 Ce^{rt} = 0$$

$$r^2 + 2\beta r + \omega_0^2 = 0$$

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

- <u>Case 2 : Critically damped motion</u> $\boxed{\beta=\omega_0}$

If $\beta=\omega_0$ then $r_1 = r_2 = -\beta$.

We only have one solution, $e^{-\beta t}$; so we need another solution. <u>Exercise</u>: $te^{-\beta t}$

Thus the general solution is

$$x(t) = (C_1 + C_2 t)\, e^{-\beta t}$$

<u>Example</u>

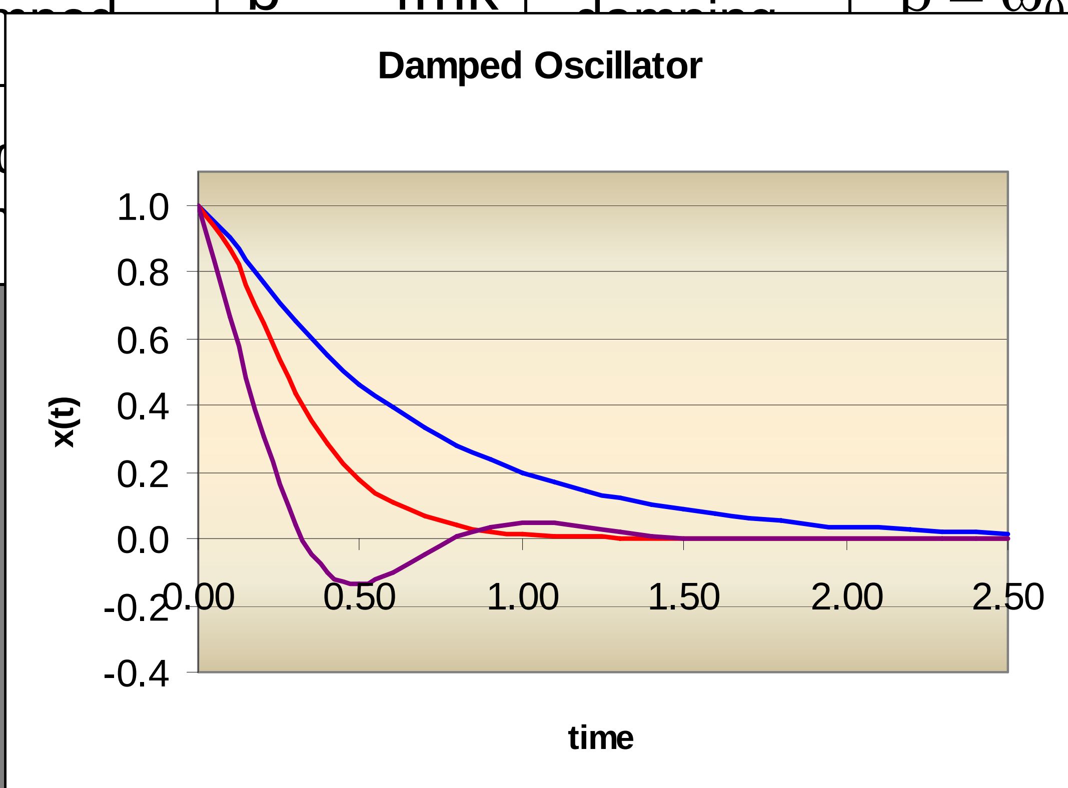
$$x(t) = A\,(1+\beta t)\, e^{-\beta t}$$

Summary

Results in terms of the original parameters

$$m\ddot{x} = -kx - b\dot{x} \qquad \begin{cases} 2\beta = \mathrm{b/m} \\ \omega_0^2 = k/m \end{cases}$$

Over-damped	$b^2 > 4mk$	strong damping	$\beta > \omega_0$
Critically dam	$b^2 = 4mk$	optimal	$\beta = \omega_0$
Un dar			

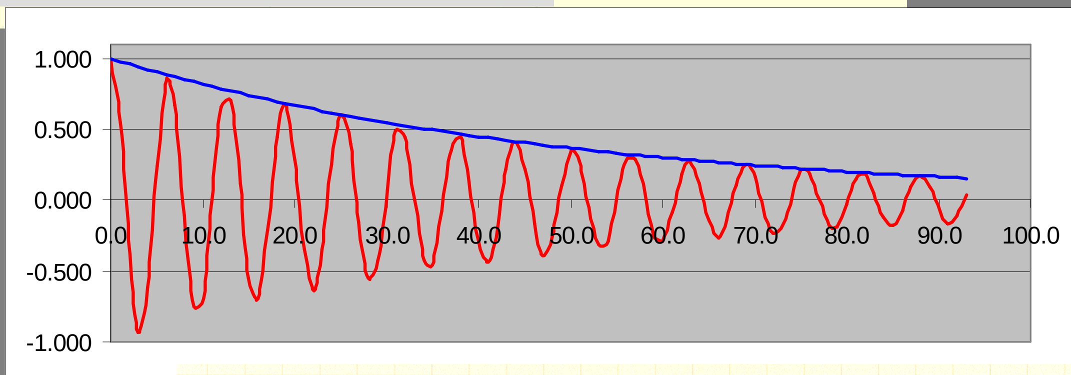


Underdamped oscillations

Consider $x(t) = A\left[\cos\omega_1 t + \frac{\beta}{\omega_1}\sin\omega_1 t\right]e^{-\beta t}$

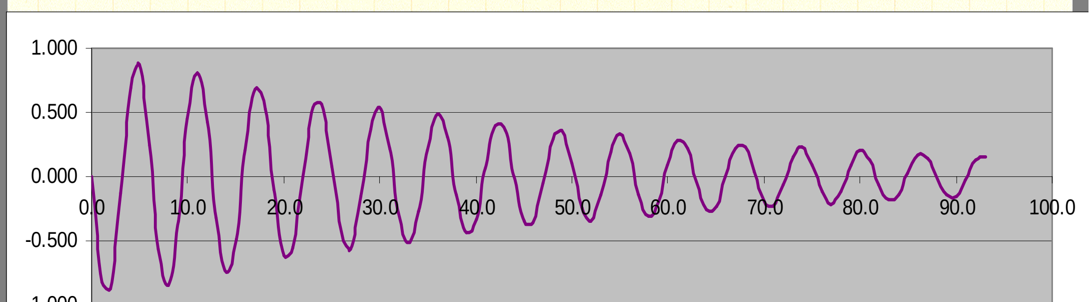
for ω_0=1.0 and β=0.02

units: s^{-1}



Velocity After a bit of algebra (EXERCISE)

$\dot{x}(t) = -A\omega_1\left[1+\left(\frac{\beta}{\omega_1}\right)^2\right]e^{-\beta t}\sin\omega_1 t$



Note that v = 0 at ω_1 t = π, 2π, 3π, 4π, ...

These are the maximum displacements from x = 0.

The maximum *positive* displacements occur at

ω_1 t = 0, 2π, 4π, 6π, …

i.e., $\omega_1 t_n = 2\pi n$ for n = 0, 1, 2, 3, 4 …

$x_{max,n} = A\exp(-\beta t_n) = A\exp(-2\pi\beta n/\omega_1)$

Energy and the underdamped oscillator

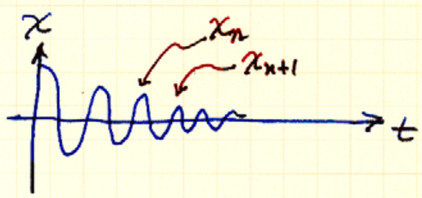
The mechanical energy is $E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$.

If $b = 0$ then E is constant.

If b is "small" then E decreases "slowly."

$$\frac{dE}{dt} = m \dot{x} \ddot{x} + k x \dot{x}$$

$$= (m \ddot{x} + kx) \dot{x} = -b \dot{x}^2$$



$$x_n = A e^{-\frac{2\pi \beta n}{\omega_1}}$$

$n = 1 \;\; 2 \;\; 3 \;\; 4, \ldots$

Comments

- $$Q = \frac{m\omega_1}{b} \quad \text{where} \quad \omega_1 = \sqrt{\omega_0^2 - (b/2m)^2}$$

$$Q = \sqrt{\left(\frac{m\omega_0}{b}\right)^2 - \frac{1}{4}}$$

For weak damping, $Q \approx \frac{m\omega_0}{b}$ and $Q \gg 1$.

- $$\frac{E_n}{E_{n+1}} = \frac{E_0 e^{-2\pi n/Q}}{E_0 e^{-2\pi(n+1)/Q}} = e^{2\pi/Q}$$

independent of n for linear damping.

Or, $$Q = \frac{2\pi}{\ln(E_n/E_{n+1})}$$

- $|\Delta E|_n = E_n - E_{n+1}$ = loss of mechanical energy from n to $n+1$

$$\frac{|\Delta E|_n}{E_n} = 1 - \frac{E_{n+1}}{E_n} = 1 - e^{-2\pi/Q}$$

↑ fraction of energy lost from n to $n+1$.

For weak damping, $2\pi/Q \ll 1$ so $\frac{|\Delta E|_n}{E_n} \approx \frac{2\pi}{Q}$ (small)

- "Quality factor": large Q means weak damping.