The Physics 431 Final Exam

Wed, DECEMBER 15, 2010
12:45 – 2:45 p.m. ☝
BPS 1320 (not 1308!!!)

- Calculators, 2 letter-size sheets “handwritten notes” OK
- Graded lab reports OK
- Books, old HW, laptops NO

The exam includes topics covered throughout the semester
Greater emphasis will be placed on the 2nd half of the course
The exam consists of problems totaling 250 pts.
Show all work on exam pages — circle your answers
Grades will be posted at BPS 4238 by 12 pm Friday, December 17. Remember your “pass code” from the final exam.

Check “Midterm Review Slides” for topics covered in Midterm I. Review “Final Exam Topics” posted/handed out in class.
Telescope

- Object is at infinity so image is at $f$
- Measure angular magnification
- Length of telescope light path is sum of focal lengths of objective and eyepiece

The exit pupil is the image of the aperture stop (AS).
Define $CA_0 =$ entrance pupil clear aperture
$CA_e =$ exit pupil clear aperture
From the diagram, it is clear that

$$M = -\frac{f_o}{f_e}$$

$$\frac{CA_0}{CA_e} = \frac{s}{s'} = \frac{\theta}{\theta'} = M.$$
Microscope

- **x'** is the tube length:
  - standard **x'** ranging 160mm to 250mm
- Magnification is product of lateral magnification of objective and angular magnification of eyepiece
- Note: Image is viewed at infinity

\[
M_0 = \frac{h'}{h} = -\frac{s_2}{s_1} = -\frac{x'}{f_0}
\]

\[
M_e = \frac{25}{f_e}
\]

\[
M_{total} = M_0 \times M_e = \frac{-x'}{f_0} \cdot \frac{25}{f_e}
\]
Eye (Hecht 5.7.1 and Notes)

The overall power of the eye is ~ 58.6 D. The lens surfaces are not spherical, and the lens index is higher at the center (on-axis). Both effects correct spherical aberration. The diameter of the iris ranges from 1.5 → 8 mm.

Topics/Keywords:
Eye model, Visual Acuity, Cones/Rods accomodation, eyeglasses, nearsightedness/myopia, farsightedness/hyperopia
Human Eye – Gullstrand Model

Table 10A  PRINCIPAL DIMENSIONS FOR GULLSTRAND’S SCHEMATIC EYE
Overall power of eye = 58.64 D

<table>
<thead>
<tr>
<th>Component</th>
<th>Refractive Index</th>
<th>Axis Position, mm</th>
<th>Radius Curvature, mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cornea, anterior and posterior</td>
<td>1.376</td>
<td>0</td>
<td>7.7</td>
</tr>
<tr>
<td>Aqueous humor</td>
<td>1.336</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vitreous humor</td>
<td>1.336</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lens:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cortex, anterior and posterior</td>
<td>1.386</td>
<td>3.6</td>
<td>10.0</td>
</tr>
<tr>
<td>Core, anterior and posterior</td>
<td>1.406</td>
<td>4.15</td>
<td>7.9</td>
</tr>
<tr>
<td>Cardinal points:</td>
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<tr>
<td>AH</td>
<td>1.348</td>
<td></td>
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</tr>
<tr>
<td>AH'</td>
<td>1.602</td>
<td></td>
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<tr>
<td>AN</td>
<td>7.08</td>
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</tr>
<tr>
<td>AN'</td>
<td>7.33</td>
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<td></td>
</tr>
<tr>
<td>AF</td>
<td>-15.70</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AF'</td>
<td>24.38</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Retina – Cones and Rods

Rods are most sensitive to light, but do not sense color, motion.

Cones are color sensitive in bright light.

You have ~ 6 million cones, ~ 120 million rods, but only 1 million nerve fibers.

Cones are 1-1.5 μm diameter, 2-2.5 μm apart in the fovea.

Rods are ~ 2 μm diameter

The macula is 5° to the outside of the axis.

The fovea is the central 0.3 mm of the macula. It has only cones and is the center of sharp vision.

Current understanding is that the 6 to 7 million cones can be divided into "red" cones (64%), "green" cones (32%), and "blue" cones (2%) based on measured response curves.
We will learn that the spatial resolution limit due to diffraction $\approx 1.22 \times f \lambda / D = 0.61 \times \lambda / NA$ [Rayleigh Criterion].
The Chief Ray

For an off-axis object, the chief ray (CR) is the ray that passes through the center of the aperture stop. Rays that pass through the edge of the aperture stop are marginal rays (MR).
The aperture stop (AS) is defined to be the stop or lens ring, which physically limits the solid angle of rays passing through the system from an on-axis object point. The aperture stop limits the brightness of an image.

The entrance pupil of a system is the image of the aperture stop as seen from an axial point on the object through those elements preceding the stop. (Hecht p. 171)

The exit pupil of a system is the image of the aperture stop as seen from an axial point on the image plane through the interposed lenses, if there is any. (Hecht p. 172)
Monochromatic plane waves

Plane waves have straight wave fronts

- As opposed to spherical waves, etc.
- Suppose

\[
E(r) = E_0 e^{i\mathbf{k} \cdot \mathbf{r}}
\]

\[
E(r, t) = \text{Re}\{E(r)e^{-i\omega t}\} = \text{Re}\{E_0 e^{i\mathbf{k} \cdot \mathbf{r}} e^{-i\omega t}\} = \text{Re}\{E_0 e^{i(k \cdot r - \omega t)}\}
\]

- \(E_0\) still contains: amplitude, polarization, phase
- Direction of propagation given by wavevector:

\[\mathbf{k} = (k_x, k_y, k_z) \text{ where } |\mathbf{k}| = 2\pi/\lambda = \omega/c\]

- Can also define

\[E = (E_x, E_y, E_z)\]

- Plane wave propagating in z-direction

\[
E(z,t) = \text{Re}\{E_0 e^{i(kz - \omega t)}\} = \frac{1}{2} \{E_0 e^{i(kz - \omega t)} + E_0^* e^{-i(kz - \omega t)}\}
\]

Key words: energy, momentum, wavelength, frequency, phase, amplitude...
Poynting vector & Intensity of Light \[ \mathbf{S} = \mathbf{E} \times \mathbf{H} \]

Summary (free space or isotropic media)

\[ \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}; \quad \| \mathbf{S} \| = c \varepsilon_0 \| \mathbf{E} \|^2 \] Poynting vec

\[ \langle \| \mathbf{S} \| \rangle = \frac{1}{T} \int_{t}^{t+T} \| \mathbf{S} \| dt \] Irradiance (or intensity)

- Poynting vector describes flows of E-M power
- Power flow is directed along this vector
  - Usually parallel to \( \mathbf{k} \)
- Intensity is equal to the magnitude of the time averaged Poyning vector: \( I = \langle \mathbf{S} \rangle \)

\[ \langle \mathbf{S} \rangle = I \equiv \langle \mathbf{E}(t) \times \mathbf{H}(t) \rangle = \frac{c \varepsilon_0}{2} \mathbf{E}^2 = \frac{c \varepsilon_0}{2} \left( E_x^2 + E_y^2 \right) \]

\[ c \varepsilon_0 \approx 2.654 \times 10^{-3} \ \text{A/}\text{V} \]

\[ \hbar \omega[eV] = \frac{1239.85}{\lambda[\text{nm}]} \]

\[ \hbar = 1.05457266 \times 10^{-34} \ \text{Js} \]

\[ \mathbf{E} = 1 \text{V/m} \]

\[ I = ? \text{W/m}^2 \]
Wave equations in a medium

The induced polarization in Maxwell’s Equations yields another term in the wave equation:

\[
\frac{\partial^2 E}{\partial z^2} - \mu \varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \quad \frac{\partial^2 E}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 E}{\partial t^2} = 0
\]

This is the **Inhomogeneous Wave Equation**.

The polarization is the driving term for a new solution to this equation.

\[
\frac{\partial^2 E}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \quad \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0
\]

**Homogeneous (Vacuum) Wave Equation**

\[
E(z, t) = \text{Re}\{E_0 e^{i(kz - \omega t)}\} = \frac{1}{2} \{E_0 e^{i(kz - \omega t)} + E_0^* e^{-i(kz - \omega t)}\} = |E_0| \cos(kz - \omega t)
\]

\[
\frac{c}{n} = v \quad \text{Phase velocity}
\]

*Phase velocity can exceed the speed of light in a dispersive medium where the refractive index $n$ is not necessarily $>1$.\]
Spherical waves

A spherical wave is also a solution to Maxwell's equations and is a good model for the light scattered by a molecule.

Unlike a plane wave, whose amplitude remains constant as it propagates, a spherical wave weakens. Its irradiance goes as \( 1/r^2 \).

\[
E(\vec{r}, t) \propto \left( \frac{E_0}{r} \right) \text{Re}\{\exp[i(kr - \omega t)]\}
\]

- where \( k \) is a scalar, and
- \( r \) is the radial magnitude.

Note that \( k \) and \( r \) are not vectors here!
Interference [Hecht 9.1-9.4, 9.7.2; Fowles 3.1-3.1; Notes]

\[ E(\mathbf{r}) = E_0 e^{ik \cdot \mathbf{r}} \]

\[ E(\mathbf{r}, t) = \text{Re}\{E(\mathbf{r})e^{-i\omega t}\} = \text{Re}\{E_0 e^{ik \cdot \mathbf{r}} e^{-i\omega t}\} = \text{Re}\{E_0 e^{i(k \cdot \mathbf{r} - \omega t)}\} \]

Consider the Optical Path Difference (OPD)
Or simply the superposition of two plane waves

\[ E(\mathbf{r}) = E_1 e^{ik_1 \cdot \mathbf{r}} + E_2 e^{ik_2 \cdot \mathbf{r}} \]

\[ I = |E(\mathbf{r})|^2 = E \times E^* \]

Key words/Topics:
Michelson Interferometer, Dielectric thin film, Anti-reflection coating, Fringes of equal thickness, Newton rings.
The Michelson Interferometer and Spatial Fringes

- Suppose we misalign the mirrors so the beams cross at an angle when they recombine at the beam splitter. And we won't scan the delay.

- If the input beam is a plane wave, the cross term becomes:

\[
\text{Re}\left\{E_0 \exp[i(\omega t - k z \cos \theta - k x \sin \theta)]E_0^* \exp[-i(\omega t - k z \cos \theta + k x \sin \theta)]\right\}
\]

\[\propto \text{Re}\{\exp[-2ikx \sin \theta]\}\]

\[\propto \cos(2kx \sin \theta)\]

Crossing beams maps delay onto position.
Suppose we change one arm’s path length.

\[
\text{Re}\left\{ E_0 \exp\left[ i(\omega t - kz \cos \theta - kx \sin \theta + 2kd) \right] E_0^* \exp\left[ -i(\omega t - kz \cos \theta + kx \sin \theta) \right]\right\}
\]

\[\propto \text{Re}\left\{ \exp\left[ -2ikx \sin \theta + 2kd \right] \right\}\]

\[\propto \cos(2kx \sin \theta + 2kd)\]

The fringes will shift in phase by \(2kd\).
The Unbalanced Michelson Interferometer can sensitively measure phase vs. position.

- Phase variations of a small fraction of a wavelength can be measured.

Placing an object in one arm of a misaligned Michelson interferometer will distort the spatial fringes.

See HW#8 Problem #1
Michelson interferometers: the compensator plate

If reflection occurs off the front surface of beam splitter, the transmitted beam passes through beam splitter three times; the reflected beam passes through only once.

So a **compensator plate** (identical to the beam splitter) is usually added to equalize the path length through glass.
Newton's Rings

From the figure, if $R > d$, then

$$x^2 R^2 - (R-d)^2 = x^2 = 2Rd$$

The interference maximum will occur if

$$2n\lambda_m = (m + \frac{1}{2})\lambda_0$$

Thus, the radius of the bright rings are

$$x_n = \sqrt{(m + \frac{1}{2})\lambda R}$$

Similarly, the radius of dark rings are

$$x_n = \sqrt{m\lambda R}$$

**Figure 9.17** Fringes of equal inclination.
Phase shift on reflection at an interface

Near-normal incidence

\(\pi\) phase shift if \(n_i < n_t\)

0 (or \(2\pi\) phase shift) if \(n_i > n_t\)

\[
\begin{align*}
    r_\perp &= \left( \frac{E_{0r}}{E_{0l}} \right)_\perp = \frac{n_i \cos \theta_i - n_t \cos \theta_t}{n_i \cos \theta_i + n_t \cos \theta_t} \\
    t_\perp &= \left( \frac{E_{0r}}{E_{0l}} \right)_\perp = \frac{2n_i \cos \theta_i}{n_i \cos \theta_i + n_t \cos \theta_t} \\
    r_\parallel &= \left( \frac{E_{0r}}{E_{0l}} \right)_\parallel = \frac{n_t \cos \theta_i - n_i \cos \theta_t}{n_t \cos \theta_i + n_i \cos \theta_t} \\
    t_\parallel &= \left( \frac{E_{0r}}{E_{0l}} \right)_\parallel = \frac{2n_t \cos \theta_i}{n_t \cos \theta_i + n_i \cos \theta_t}
\end{align*}
\]

Note: independent of polarization

\[
\begin{align*}
    r_\perp &= r_\parallel = \frac{n_t - n_i}{n_t + n_i} \\
    t_\perp &= t_\parallel = \frac{2n_i}{n_t + n_i}
\end{align*}
\]

I. Transmission and reflection at a boundary

The sketches below show a pulse approaching a boundary between two springs. In one case, the pulse approaches the boundary from the left; in the other, from the right. The springs are the same in both cases, and the linear mass density is greater for the spring on the right than for the spring on the left.

Before:

After:

Complete the sketches to show the shape of the springs a short time after the trailing edge of the pulse shown has reached the boundary. Be sure to show correctly (1) the relative widths of the pulses and (2) which side of the spring each pulse is on. (Ignore relative amplitudes.)
order $m$ maxima occur at:

$$m \lambda \approx a \sin \theta_m \approx a \frac{y_m}{s}$$
Diffraction

Fresnel approximation

Huygens-Fresnel integral in rectangular coordinates:

Farther out in $z$, we can approximate the quadratic phase as flat

$$ z \approx \frac{b(z^2 + \eta^2)_{\text{max}}}{2} $$

This region is referred to as the “far-field” or Fraunhofer region.

$$ U(x, y) = \frac{e^{jkz}}{j\lambda z} \int \int_{-\infty}^{\infty} dx dx d\eta d\xi U(\xi, \eta) \exp \left\{ -j \frac{2\pi}{\lambda z} (x\xi + y\eta) \right\} $$

$$ \mathcal{F}\{U(\xi, \eta)\} \bigg|_{x} = \frac{x}{\lambda z}, \quad \mathcal{F}\{U(\xi, \eta)\} \bigg|_{y} = \frac{y}{\lambda z} $$

Now this is exactly the Fourier transform of the aperture distribution with

$$ f_x = \frac{x}{\lambda z}, \quad f_y = \frac{y}{\lambda z} $$

The Fraunhofer region is farther out. For the field size of 1 cm, and $\lambda = 0.5 \mu m$, we find the valid range of $z \approx 150$ meters!

Again, examining the full integral, Fraunhofer is actually accurate and usable to much closer distances.

(A)

Let’s examine the validity of the Fresnel approximation in the Fresnel integral. The next higher order term in exponent must be small compared to 1. So the valid range of the Fresnel approximation is:

$$ z^3 \approx \frac{\pi}{4\lambda} [(x-\xi)^2 + (y-\eta)^2]_{\text{max}}^2 $$

For field sizes of 1 cm, $\lambda = 0.5 \mu m$, we find $z \approx 25$ cm.

Actually we should look at the effect on the total integral. Upon closer analysis, it is found that the Fresnel approximation holds for a much closer $z$. This is referred to as the “near-field region”.
We wish to find the light electric field after a screen with a hole in it. This is a very general problem with far-reaching applications.

What is $E(x_1,y_1)$ at a distance $z$ from the plane of the aperture?
The field in the observation plane, \( E(x_1, y_1) \), at a distance \( z \) from the aperture plane is given by:

\[
E(x_1, y_1, z) = \int \int_{A(x_0, y_0)} h(x_1 - x_0, y_1 - y_0, z) E(x_0, y_0) \, dx_0 \, dy_0
\]

where:

\[
h(x_1 - x_0, y_1 - y_0, z) = \frac{1}{i\lambda} \frac{\exp(ikr_{01})}{r_{01}}
\]

and:

\[
r_{01} = \sqrt{z^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2}
\]

A very complicated result! And we cannot approximate \( r_{01} \) in the exp by \( z \) because it gets multiplied by \( k \), which is big, so relatively small changes in \( r_{01} \) can make a big difference!
We can approximate \( r_{01} \) in the denominator by \( z \), and if \( D \) is the size of the aperture, \( D^2 \geq x_0^2 + y_0^2 \), so when \( k D^2/2z \ll 1 \), the quadratic terms \( \ll 1 \), so we can neglect them:

\[
\begin{align*}
  r_{01} &= \sqrt{z^2 + (x_0 - x_1)^2 + (y_0 - y_1)^2} \\
  &\approx z \left[ 1 + \frac{(x_0 - x_1)^2}{2z^2} + \frac{(y_0 - y_1)^2}{2z^2} \right]
\end{align*}
\]

\[
kr_{01} \approx kz + k \left( \frac{x_0^2 - 2x_0 x_1 + x_1^2}{2z} + \frac{y_0^2 - 2y_0 y_1 + y_1^2}{2z} \right)
\]

Small, so neglect these terms. Independent of \( x_0 \) and \( y_0 \), so factor these out.

\[
E(x_1, y_1) = \frac{\exp(ikz)}{i\lambda z} \exp \left[ \frac{ik}{2z} \left( x_1^2 + y_1^2 \right) \right] \iint_{A(x_0,y_0)} \exp \left\{ - \frac{ik}{z} (x_0 x_1 + y_0 y_1) \right\} E(x_0, y_0) \, dx_0 \, dy_0
\]

This condition means going a distance away: \( z \gg kD^2/2 = \pi D^2/\lambda \)

If \( D = 1 \text{ mm} \) and \( \lambda = 1 \text{ micron} \), then \( z \gg 3 \text{ m} \).
Fraunhofer Diffraction

We’ll neglect the phase factors, and we’ll explicitly write the aperture function in the integral:

\[
E(x_1, y_1) \propto \iint_{-\infty}^{\infty} \iint_{-\infty}^{\infty} \exp\left\{ -\frac{ik}{z} \left( x_0x_1 + y_0y_1 \right) \right\} A(x_0, y_0) E(x_0, y_0) \, dx_0 \, dy_0
\]

This is just a Fourier Transform!

\[ E(x_0, y_0) = \text{constant if a plane wave} \]

Interestingly, it’s a Fourier Transform from position, \( x_0 \), to another position variable, \( x_1 \) (in another plane). Usually, the Fourier “conjugate variables” have reciprocal units (e.g., \( t & \omega \), or \( x & k \)). The conjugate variables here are really \( x_0 \) and \( k_x = k x_1 / z \), which have reciprocal units.

So the far-field light field is the Fourier Transform of the apertured field!
Diffraction: single, double, multiple slits

Study Guide: Hecht Ch. 10.2.1-10.2.6 (detailed lengthy discussions), Fowles Ch. 5 (short but clear presentation), or Lecture Notes

\[ I(\beta) = I(0) \left( \frac{\sin \beta}{\beta} \right)^2 \]

\[ \beta = \frac{kb}{2} \sin \theta = \frac{\pi b}{\lambda} \sin \theta \]

Java applet – Single Slit Diffraction
http://www.walter-fendt.de/ph14e/singleslit.htm
Diffraction: Double and Multiple Slits

\[ I(\theta) = I(0) \left( \frac{\sin \beta}{\beta} \right)^2 \left( \frac{\sin N\gamma}{N\sin \gamma} \right)^2 \]

\[ \beta = \frac{1}{2} kb \sin \theta; \quad \gamma = \frac{1}{2} ka \sin \theta \]

See also

http://demonstrations.wolfram.com/MultipleSlitDiffractionPattern/ and

http://wyant.optics.arizona.edu/multipleSlits/multipleSlits.htm
Fraunhofer diffraction from two slits (Fourier Transform)

\[ A(x_0) = \text{rect}\left[\frac{(x_0 + a)}{w}\right] + \text{rect}\left[\frac{(x_0 - a)}{w}\right] \]

\[
E(x_1) \propto \mathcal{F}\{A(x_0)\}
\]

\[ \propto \text{sinc}\left[\frac{w(kx_1 / z)}{2}\right]\exp[+ia(kx_1 / z)] + \text{sinc}\left[\frac{w(kx_1 / z)}{2}\right]\exp[-ia(kx_1 / z)] \]

\[ E(x_1) \propto \text{sinc}\left(\frac{wkx_1}{2z}\right) \cos\left(\frac{akx_1}{z}\right) \]
Diffraction from one- and two-slit screens

Fraunhofer diffraction patterns

One slit

Two slits
Diffraction Gratings

Scattering ideas explain what happens when light impinges on a periodic array of grooves. Constructive interference occurs if the delay between adjacent beamlets is an integral number, $m$, of wavelengths.

Path difference: $AB - CD = m\lambda$

$$a \left[ \sin(\theta_m) - \sin(\theta_i) \right] = m\lambda$$

where $m$ is any integer.

A grating has solutions of zero, one, or many values of $m$, or orders.

Remember that $m$ and $\theta_m$ can be negative, too.
Because the diffraction angle depends on $\lambda$, different wavelengths are separated in the nonzero orders.

The longer the wavelength, the larger its deflection in each nonzero order.
The Diffraction Grating

Hecht 10.2.8 or Fowles Ch. 5 p.123 (handout)

Grating Equation

(Optical Path Difference OPD= \( m \lambda \))

\[
a \left( \sin \theta_m - \sin \theta_i \right) = m\lambda
\]

\[
a \sin \theta_m = m\lambda \quad \text{Normal incidence } \theta_i = 0
\]

The chromatic/spectral resolving power of a grating

\[
R \equiv \frac{\lambda}{\Delta \lambda} = mN
\]

\( m \) is the order number, and
\( N \) is the total number of gratings.
Uniform Rectangular Aperture

\[ I(\theta) = I(0) \left( \frac{\sin \alpha}{\alpha} \right)^2 \left( \frac{\sin \beta}{\beta} \right)^2 \]

\[ \alpha = \frac{1}{2} ka \sin \theta; \quad \beta = \frac{1}{2} kb \sin \theta \]
Uniform Circular Aperture

\[ I(\theta) = I(0) \left( \frac{2J_1(\rho)}{\rho} \right)^2 \]

\[ \rho = kR \sin \theta; \quad k = \frac{2\pi}{\lambda} \]

A circular aperture yields a diffracted "Airy Pattern," which involves a Bessel function.
Recall the Scale Theorem!
This is the Uncertainty Principle for diffraction.

Far-field intensity pattern from a small aperture

Far-field intensity pattern from a large aperture
Wave optics of a lens

We have previously seen that light passing through a lens experiences a phase delay given by:
\[ \varphi(x, y) = e^{-jk(n-1)\left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} \] (neglecting the constant phase)

The focal length, \( f \) is given by:
\[ \frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \] The “lens makers formula”

The transmission function is now:
\[ \varphi(x, y) = e^{-\frac{jk}{2f}(x^2+y^2)} \]

This is the paraxial approximation to the spherical phase

Note: the incident plane-wave is converted to a spherical wave converging to a point at \( f \) behind the lens (\( f \) positive) or diverging from the point at \( f \) in front of lens (\( f \) negative).

The focal plane amplitude distribution is a Fourier transform of the lens pupil function \( P(x,y) \), multiplied by a quadratic phase term. However, the intensity distribution is exactly

\[ I_f(u, v) = \frac{2}{\lambda f} |\mathcal{F}[P(x,y)]|^2 \]

\[ f_x = \frac{u}{\lambda f} \]

\[ f_y = \frac{v}{\lambda f} \]

Example: a circular lens, with radius \( w \)

The spot diameter is
\[ d = 1.22 \frac{\lambda f}{w} = 1.22 \frac{\lambda}{\theta} \]

The resolution of the lens as defined by the “Rayleigh” criterion is
\[ \frac{d}{2} = 0.61\frac{\lambda}{\theta} \]

For a small angle \( \theta \),
\[ \frac{d}{2} = 0.61\frac{\lambda}{\sin \theta} = 0.61 \frac{\lambda}{NA} \]
Gaussian Beam Optics (only eq. (4))

\[ I(r) = I_0 e^{-2r^2/w^2} = \frac{2P}{\pi w^2} e^{-2r^2/w^2} \]

\[ R(z) = z \left[ 1 + \left( \frac{\pi w_0^2}{\lambda z} \right)^2 \right] \]

and

\[ w(z) = w_0 \left[ 1 + \left( \frac{\lambda z}{\pi w_0} \right)^2 \right]^{1/2} = w_0 \left[ 1 + \left( \frac{z}{z_R} \right)^2 \right]^{1/2} \]

where we have defined a new parameter, called the Rayleigh range,

\[ z_R = \frac{\pi w_0^2}{\lambda}, \]

which combines the wavelength and waist radius into a single parameter and completely describes the divergence of the Gaussian beam. Note that the Rayleigh range is the distance from the beam waist to the point at which the beam radius has increased to \( \sqrt{2} w_0 \). For a 633 nm red He-Ne laser with a waist of 0.4 mm, \( z_R = 0.8 \) m.

When \( z \gg z_R \), Eq. (2) simplifies to \( w = w_0 z/z_R \) and the laser beam diverges at a constant angle

\[ \theta = \frac{W}{z} = \frac{w_0}{z_R} = \frac{\lambda}{\pi w_0} \]

Note that the smaller the Rayleigh range, the more rapidly the beam diverges.
Basic Fourier Optics (~30-50 points)
There are several ways to denote the Fourier transform of a function.

If the function is labeled by a lower-case letter, such as $f$, we can write:

$$f(t) \rightarrow F(\omega)$$

If the function is already labeled by an upper-case letter, such as $E$, we can write:

$$E(t) \rightarrow \mathcal{F} \{ E(t) \} \quad \text{or:} \quad E(t) \rightarrow \tilde{E}(\omega)$$

Sometimes, this symbol is used instead of the arrow:
Example: the Fourier Transform of a rectangle function: \( \text{rect}(t) \)

\[
F(\omega) = \int_{-1/2}^{1/2} \exp(-i\omega t)\,dt = \frac{1}{-i\omega} \left[ \exp(-i\omega t) \right]_{-1/2}^{1/2} \\
= \frac{1}{-i\omega} \left[ \exp(-i\omega/2) - \exp(i\omega/2) \right] \\
= \frac{1}{(\omega/2)} \frac{\exp(i\omega/2) - \exp(-i\omega/2)}{2i} \\
= \frac{\sin(\omega/2)}{(\omega/2)}
\]

\[
F(\omega) = \text{sinc}(\omega/2)
\]
The Fourier Transform of $\delta(t)$ is 1.

$$\int_{-\infty}^{\infty} \delta(t) \exp(-i\omega t) \, dt = \exp(-i\omega[0]) = 1$$

And the Fourier Transform of 1 is $2\pi \delta(\omega)$:

$$\int_{-\infty}^{\infty} 1 \exp(-i\omega t) \, dt = 2\pi \delta(\omega)$$
The Fourier transform of \( \exp(i \omega_0 t) \)

\[
\mathcal{F}\{ \exp(i \omega_0 t) \} = \int_{-\infty}^{\infty} \exp(i \omega_0 t) \exp(-i \omega t) \, dt \\
= \int_{-\infty}^{\infty} \exp(-i[\omega - \omega_0]t) \, dt = 2\pi \delta(\omega - \omega_0)
\]

The function \( \exp(i \omega_0 t) \) is the essential component of Fourier analysis. It is a pure frequency.
The Fourier transform of $\cos(\omega_0 t)$

\[
\mathcal{F}\{\cos(\omega_0 t)\} = \int_{-\infty}^{\infty} \cos(\omega_0 t) \exp(-i \omega t) \, dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} [\exp(i \omega_0 t) + \exp(-i \omega_0 t)] \exp(-i \omega t) \, dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} \exp(-i[\omega - \omega_0]t) \, dt + \frac{1}{2} \int_{-\infty}^{\infty} \exp(-i[\omega + \omega_0]t) \, dt
\]

\[
= \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)
\]
The Fourier transform of a scaled function, $f(at)$:

$$\mathcal{F}\left\{f(at)\right\} = F(\omega/a) / \left|a\right|$$

Proof:

$$\mathcal{F}\left\{f(at)\right\} = \int_{-\infty}^{\infty} f(at) \exp(-i\omega t) \, dt$$

Assuming $a > 0$, change variables: $u = at$

$$\mathcal{F}\left\{f(at)\right\} = \int_{-\infty}^{\infty} f(u) \exp(-i\omega [u/a]) \, du / a$$

$$= \int_{-\infty}^{\infty} f(u) \exp(-i [\omega/a] u) \, du / a$$

$$= F(\omega/a) / a$$

If $a < 0$, the limits flip when we change variables, introducing a minus sign, hence the absolute value.
The Scale Theorem in action

The shorter the pulse, the broader the spectrum!

This is the essence of the Uncertainty Principle!
The Fourier Transform of a sum of two functions

\[ \mathcal{F}\{a f(t) + b g(t)\} = a \mathcal{F}\{f(t)\} + b \mathcal{F}\{g(t)\} \]

Also, constants factor out.
The Fourier transform of a shifted function, $f(t - a)$:

$$\mathcal{F} \left\{ f(t - a) \right\} = \exp(-i\omega a)F(\omega)$$

Proof:

$$\mathcal{F} \left\{ f(t - a) \right\} = \int_{-\infty}^{\infty} f(t - a) \exp(-i\omega t) dt$$

Change variables: $u = t - a$

$$\int_{-\infty}^{\infty} f(u) \exp(-i\omega[u + a]) du$$

$$= \exp(-i\omega a) \int_{-\infty}^{\infty} f(u) \exp(-i\omega u) du$$

$$= \exp(-i\omega a)F(\omega)$$
Fourier Transform with respect to space

If \( f(x) \) is a function of position,\[ F(k) = \int_{-\infty}^{\infty} f(x) \exp(-ikx) \, dx \]
\[ \mathcal{F}\{f(x)\} = F(k) \]

We refer to \( k \) as the **spatial frequency**.

Everything we’ve said about Fourier transforms between the \( t \) and \( \omega \) domains also applies to the \( x \) and \( k \) domains.
The 2D Fourier Transform

\[ \mathcal{F}^{(2)} \{ f(x,y) \} = F(k_x, k_y) \]

\[ = \int \int f(x,y) \exp[-i(k_x x + k_y y)] \, dx \, dy \]

If \( f(x,y) = f_x(x) f_y(y) \),

then the 2D FT splits into two 1D FT's.

But this doesn't always happen.
Fibers (will not be covered in 2010)

1. Total reflection.
2. Corning Glass Works, 1970: fiber with similar attenuation of copper cable. 1% per km, or 20 dB/km. Currently, 96% per km or better, i.e., 0.16 dB/km.
3. Benefit comparing to copper cables: low-loss, high data rate, small size and weight, immune to electromagnetic interference, low cost.
4. Calculation of acceptance angle \( \theta_{\text{max}} \), which is the maximum incident angle for a ray to experience total reflection in the fiber.

\[
\theta_c = \frac{n_c}{n_f} \sin(90^\circ - \theta_f)
\]

Thus,

\[
\frac{n_c}{n_f} = \cos \theta_f = \sqrt{1 - \sin^2 \theta_f}
\]

Applying Snell's Law,

\[
\sin \theta_{\text{max}} = \frac{1}{n_f} \sqrt{n_f^2 - n_c^2}
\]

Numerical aperture (NA): \( n_f \sin \theta_{\text{max}} \), the light-gathering power.

\[
NA = \left( n_f^2 - n_c^2 \right)^{1/2}
\]

Example:

Let axial length be \( L \), the shortest length of ray path. Then, the longest path \( L_{\text{max}} \) is when the incident angle is \( \theta_c \). The time difference \( \Delta t \) becomes

\[
\Delta t = \frac{L}{v_f} - \frac{L_{n_f}}{c} = \frac{L_{n_f}}{c} \left( \frac{n_f}{n_c} - 1 \right)
\]

If \( n_f = 1.5 \) and \( n_c = 1.489 \), then \( \Delta t/L = 37 \) ns/km, or a separation of distance 7.4 m/km. In order to make the signal readable, the spatial separation might need to be twice of the spread-out width. If the line is 1 km long, the output pulse is 7.4 m long, the separation should be 14.8 m or 74 ns apart, which is 13.5 Million/s.

The number of modes in a stepped-index fiber is

\[
N_m \approx \frac{1}{2} \left( \pi D \times \frac{\text{NA}}{\lambda_0} \right)^2
\]