The Physics 431 Final Exam

WED, DECEMBER 15, 2010

12:45 - 2:45 P.M.

BPS 1320 (NOT 1308!!!)

- **Calculators, 2 letter-size sheets “handwritten notes” OK**
- **Graded lab reports ================→ OK**
- **Books, old HW, laptops NO**

The exam includes topics covered throughout the semester

Greater emphasis will be placed on the 2nd half of the course

The exam consists of problems totaling 250 pts.

Show all work on exam pages — circle your answers

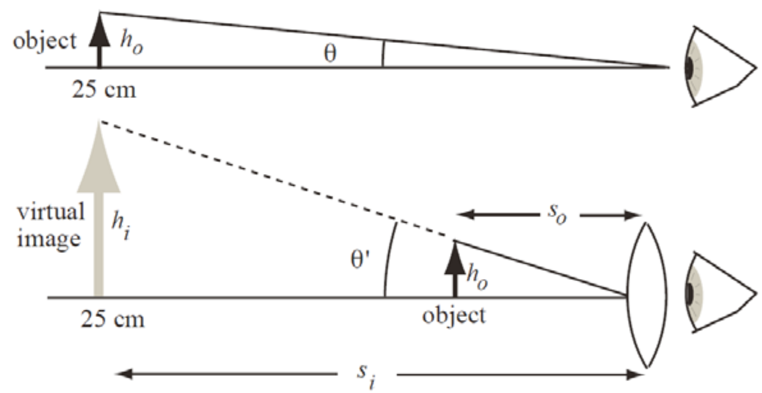
Grades will be posted at BPS 4238 by 12 pm **Friday, December 17. Remember your “pass code” from the final exam.**

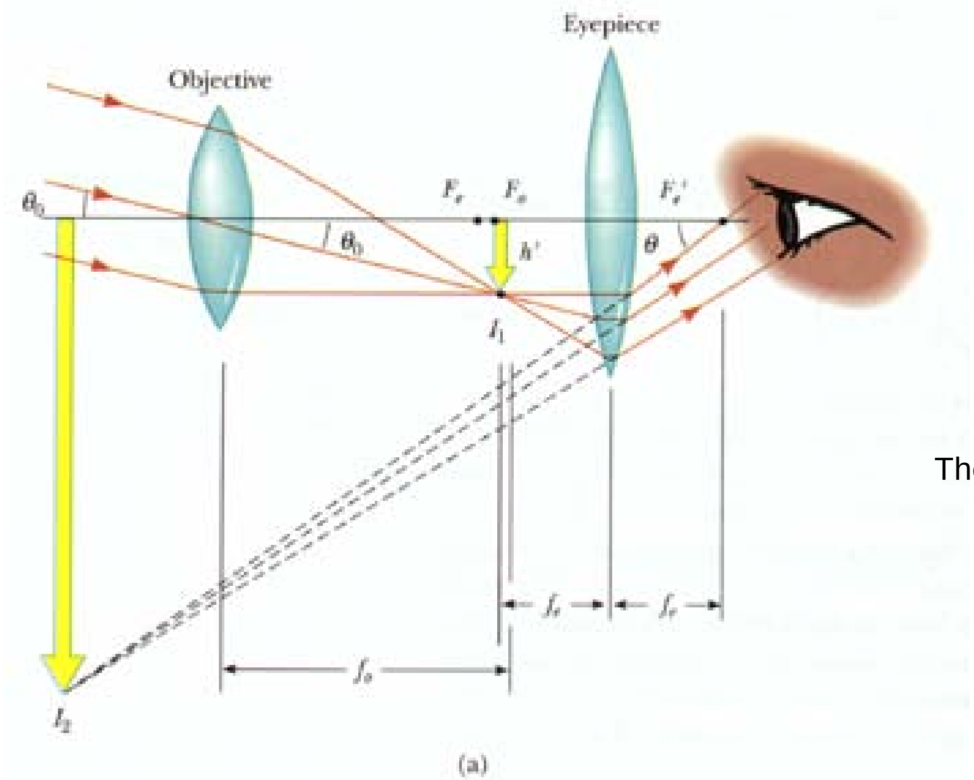
Check “Midterm Review Slides” for topics covered in Midterm I.
Review “Final Exam Topics” posted/handed out in class.

Telescope

- Object is at infinity so image is at f
- Measure angular magnification
- Length of telescope light path is sum of focal lengths of objective and eyepiece

A Magnifying Lens





$$M = -\frac{f_o}{f_e}$$

$$\frac{CA_0}{CA_e} = \frac{s}{s'} = \frac{\theta'}{\theta} = M.$$

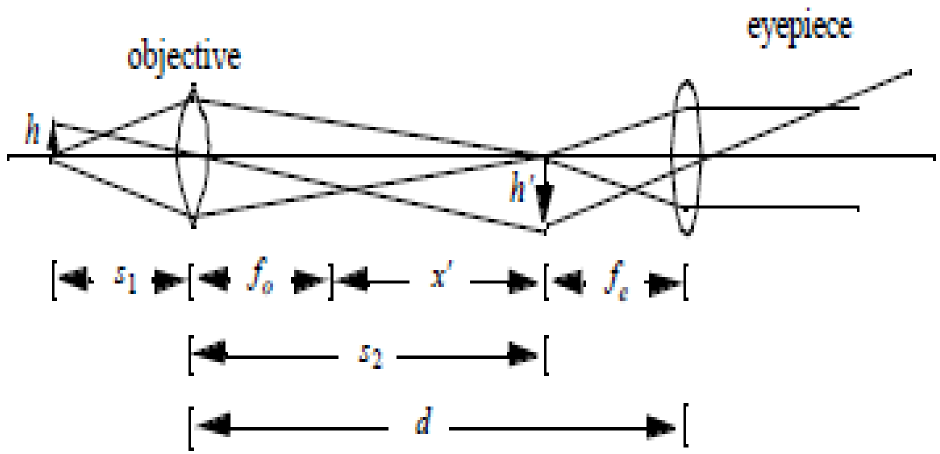
The **exit pupil** is the image of the aperture stop (AS).

Define CA_0 = entrance pupil clear aperture

CA_e= exit pupil clear aperture

From the diagram, it is clear that

Microscope



- The objective lens produces a real (inverted), magnified image of the object.
- The eyepiece re-images to a comfortable viewing distance and provides additional magnification.

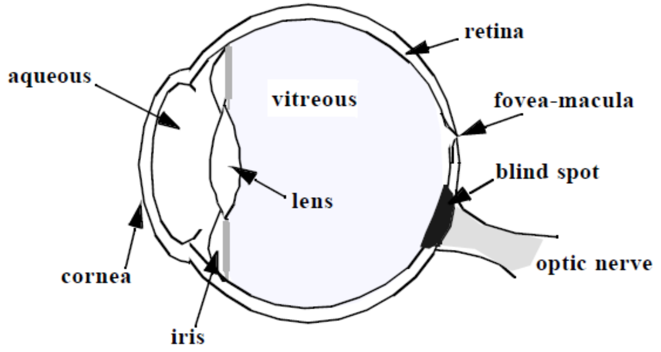
- x‘ is the tube length:
  standard x’ ranging160mm to 250mm
- Magnification is product of lateral magnification of objective and angular magnification of eyepiece
- Note: Image is viewed at infinity

$$M_0 = \frac{h^{'}}{h} = -\frac{s_2}{s_1} = \frac{-x^{'}}{f_0}$$

$$M_e = \frac{25}{f_e}$$

$$M_{total} = M_0 \times M_e = \frac{-x^{'}}{f_0} \cdot \frac{25}{f_e}$$

Eye (Hecht 5.7.1 and Notes)



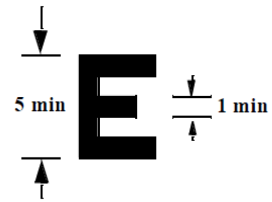
	t (mm)	n	R (mm)
cornea f	0	1.376	7.7
b	0.5		6.8
aqueous		1.336	
lens f	4.0	1.386-1.406	10.0 (relaxed), 5 (focused)
b	7.0		–6.0 (relaxed), –5 (focused)
vitreous		1.336	
retina	24.4		

Visual Acuity (VA)

The separation between cone cells in the fovea corresponds to about $1'$ (0.3 mrad). At close viewing distance of 25 cm, this gives a resolution of $75\ \mu\text{m}$.

This is close to the diffraction limit imposed by NA of the eye.

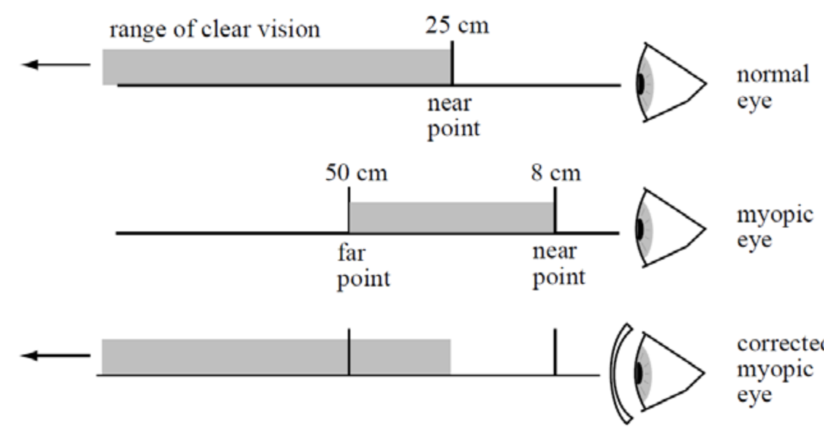
Visual acuity (VA) is defined relative to a standard of 1 minute of arc. VA = 1/(the angular size of smallest element of a letter that can be distinguished [in min])



VA is usually expressed as $\boxed{\dfrac{\text{dist to target (usually 20 ft)}}{\text{dist at which target element is 1 min}}}$

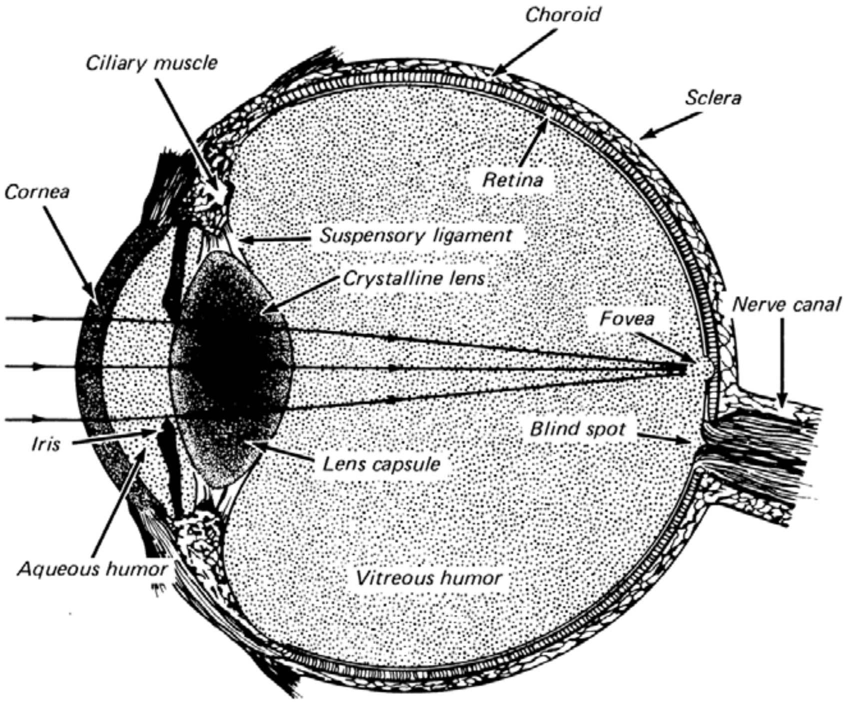
For 20/20 vision, the minimum element is 1 min at 20 ft.

The overall power of the eye is ~ 58.6 D. The lens surfaces are not spherical, and the lens index is higher at the center (on-axis). Both effects correct spherical aberration. The diameter of the iris ranges from 1.5 $\rightarrow$ 8 mm.



Topics/Keywords:
Eye model, Visual Acuity, Cones/Rods accomodation, eyeglasses, nearsightedness/myopia, farsightedness/hyperopia

Human Eye – Gullstrand Model



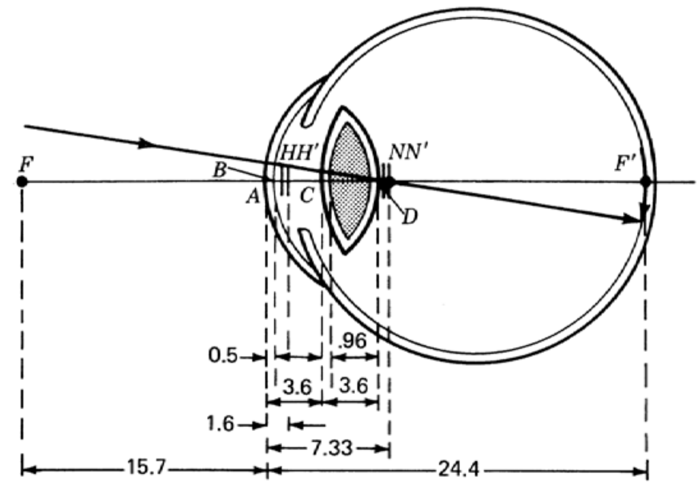


Table 10A PRINCIPAL DIMENSIONS FOR GULLSTRAND'S SCHEMATIC EYE
Overall power of eye = 58.64 D

	Refractive index	Axis position, mm	Radius curvature, mm
Cornea, anterior	1.376	0	7.7
and posterior		0.5	6.8
Aqueous humor	1.336		
Vitreous humor	1.336		
Lens:			
Cortex, anterior	1.386	3.6	10.0
and posterior		7.2	−6.0
Core, anterior	1.406	4.15	7.9
and posterior		6.57	5.8
Cardinal points:			
AH		1.348	
AH'		1.602	
AN		7.08	
AN'		7.33	
AF		−15.70	
AF'		24.38	

Retina – Cones and Rods

Rods are most sensitive to light, but do not sense color, motion

Cones are color sensitive in bright light.

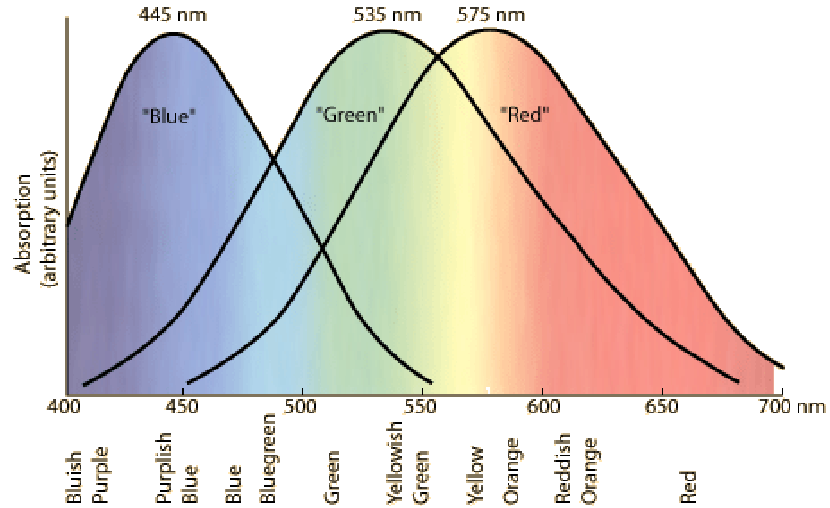
You have ~ 6 million cones, ~ 120 million rods, but only 1 million nerve fibers.

Cones are 1 -1.5 µm diameter, 2 –2.5 µm apart in the fovea.

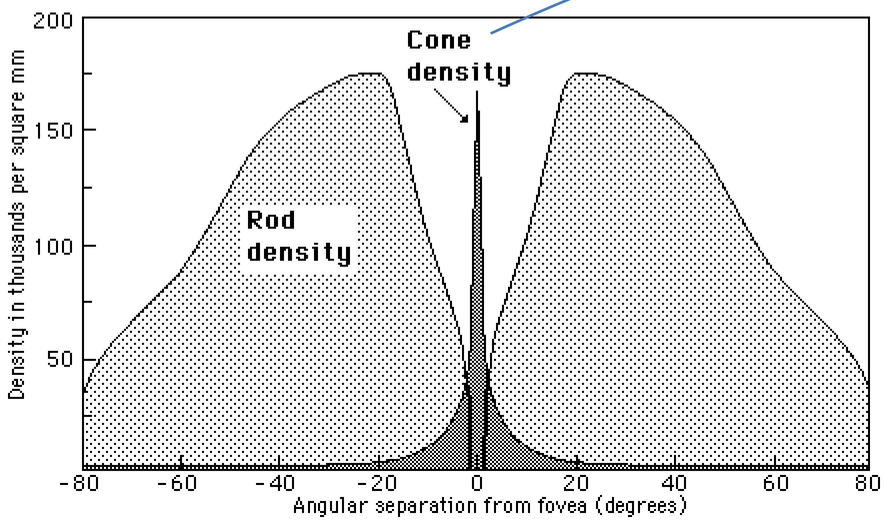
Rods are ~ 2 µm diameter

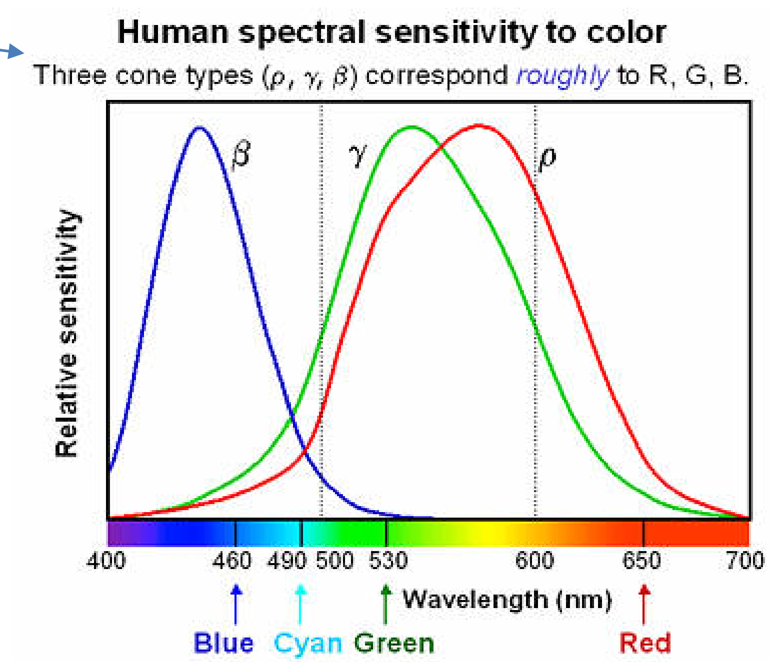
The macula is 5° to the outside of the axis.

The fovea is the central 0.3 mm of the macula. It has **only cones** and is the center of sharp vision.

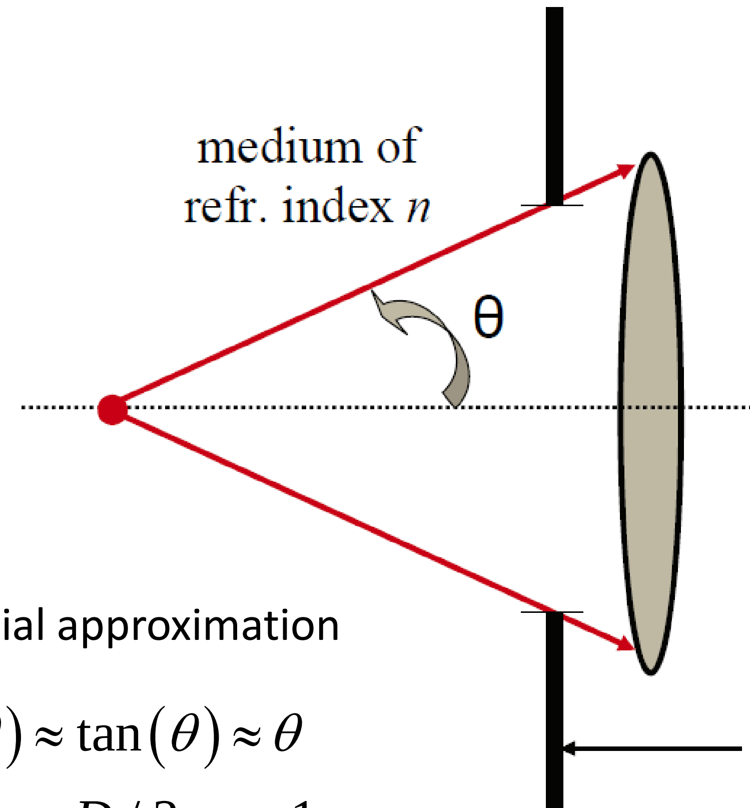


Current understanding is that the 6 to 7 million cones can be divided into "red" cones (64%), "green" cones (32%), and "blue" cones (2%) based on measured response curves.





Numerical Aperture



θ: half-angle subtended by the imaging system from an *axial* object

Numerical Aperture (NA) = $n \sin\theta$

Speed (f/#)=1/2(NA)
pronounced f-number, e.g. f/8 means (f/#)=8.

Aperture stop
the physical element which limits the angle of acceptance of the imaging system

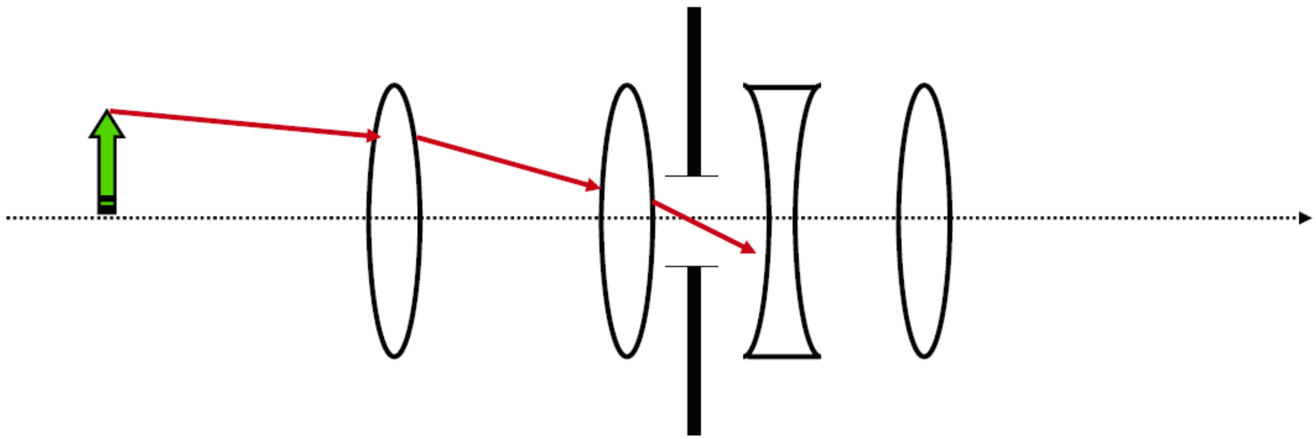
Paraxial approximation

$$\sin(\theta) \approx \tan(\theta) \approx \theta$$

$$\rightarrow NA = \frac{D/2}{f} = \frac{1}{2f/\#}$$

We will learn that
the spatial resolution limit due to diffraction ≈ 1.22×f λ /D=0.61×λ/NA [Rayleigh Criterion].

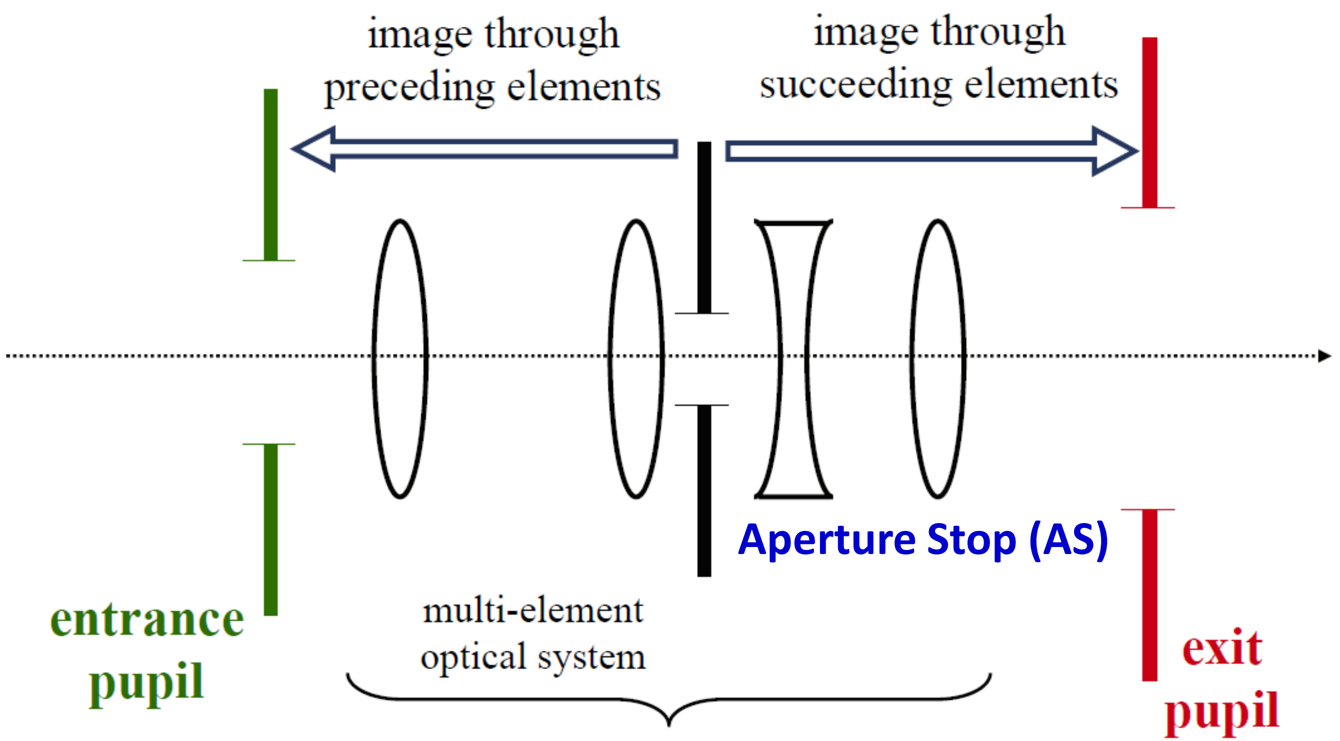
The Chief Ray



Starts from off-axis object,
Goes through the center of the Aperture

For an off-axis object, the chief ray (CR) is the ray that passes through the center of the aperture stop. Rays that pass through the edge of the aperture stop are marginal rays (MR).

Aperture Stop and Entrance & Exit Pupil



The **aperture stop** (AS) is defined to be the stop or lens ring, which physically limits the solid angle of rays passing through the system from an **on-axis** object point. The aperture stop limits the brightness of an image.

The **entrance pupil** of a system is **the image of the aperture stop as seen from an axial point on the object through those elements preceding the stop**. (Hecht p. 171)
The **exit pupil** of a system is **the image of the aperture stop as seen from an axial point on the image plane through the interposed lenses, if there is any**. (Hecht p. 172)

Monochromatic plane waves

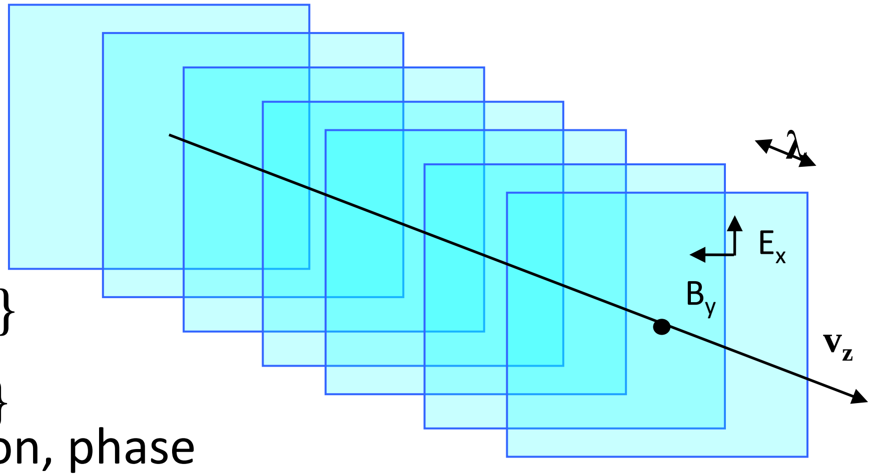
Plane waves have straight wave fronts

- As opposed to spherical waves, etc.
- Suppose

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$$

➔

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r})e^{-i\omega t}\}$$
$$= \mathrm{Re}\{\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}\}$$
$$= \mathrm{Re}\{\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\}$$



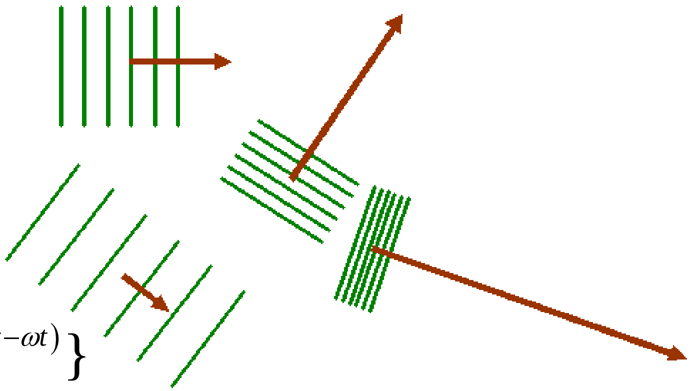
- $\mathbf{E}_0$ still contains: amplitude, polarization, phase
- Direction of propagation given by wavevector:

$\mathbf{k} = (k_x, k_y, k_z)$ where $|\mathbf{k}| = 2\pi/\lambda = \omega/\mathrm{c}$

- Can also define

$\mathbf{E} = (E_x, E_y, E_z)$

- Plane wave propagating in z-direction



$$\mathbf{E}(z,t) = \mathrm{Re}\{\mathbf{E}_0 e^{i(kz-\omega t)}\} = \tfrac{1}{2}\{\mathbf{E}_0 e^{i(kz-\omega t)} + \mathbf{E}_0^* e^{-i(kz-\omega t)}\}$$

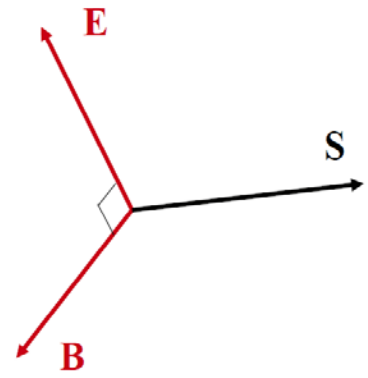
Key words: energy, momentum, wavelength, frequency, phase, amplitude…

Poynting vector & Intensity of Light $\mathbf{S}=\mathbf{E}\times\mathbf{H}$

Summary (free space or isotropic media)

$$\mathbf{S}=\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B};\quad \|\mathbf{S}\|=c\varepsilon_0\|\mathbf{E}\|^2$$ Poynting vec

$$\langle\|\mathbf{S}\|\rangle=\frac{1}{T}\int_t^{t+T}\|\mathbf{S}\|\mathrm{d}t$$ Irradiance (or intensity)



$$\mathbf{S}=\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}=c^2\varepsilon_0\mathbf{E}\times\mathbf{B}$$

so in free space

$$\mathbf{S}\,\|\,\mathbf{k}$$

S has units of W/m^2 so it represents energy flux (energy per unit time & unit area)

- Poynting vector describes flows of E-M power
- Power flow is directed along this vector
  - Usually parallel to **k**
- Intensity is equal to the magnitude of the time averaged Poyning vector: I=<**S**>

$$|\langle\mathbf{S}\rangle|=I\equiv|\langle\mathbf{E}(t)\times\mathbf{H}(t)\rangle|=\frac{c\varepsilon_0}{2}E^2=\frac{c\varepsilon_0}{2}\left(E_x{}^2+E_y{}^2\right)$$

$$c\varepsilon_0\approx 2.654\times10^{-3}\,A/V$$

example

$$E=1V/m$$

$$I=?\,W/m^2$$

$$\hbar\omega[eV]=\frac{1239.85}{\lambda[nm]}$$

$$\hbar=1.05457266\times10^{-34}\,Js$$

Wave equations in a medium

The induced polarization in Maxwell's Equations yields another term in the wave equation:

$$\frac{\partial^2 E}{\partial z^2} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \qquad \frac{\partial^2 E}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0$$

This is the **Inhomogeneous Wave Equation**.

The polarization is the driving term for a new solution to this equation.

$$\frac{\partial^2 E}{\partial z^2} - \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0 \qquad \frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Homogeneous (Vacuum) Wave Equation

$$\begin{aligned}\mathbf{E}(z,t) &= \mathrm{Re}\{\mathbf{E}_0 e^{i(kz-\omega t)}\} \\ &= \tfrac{1}{2}\{\mathbf{E}_0 e^{i(kz-\omega t)} + \mathbf{E}_0^* e^{-i(kz-\omega t)}\} \\ &= |\mathbf{E}_0| \cos(kz-\omega t)\end{aligned}$$

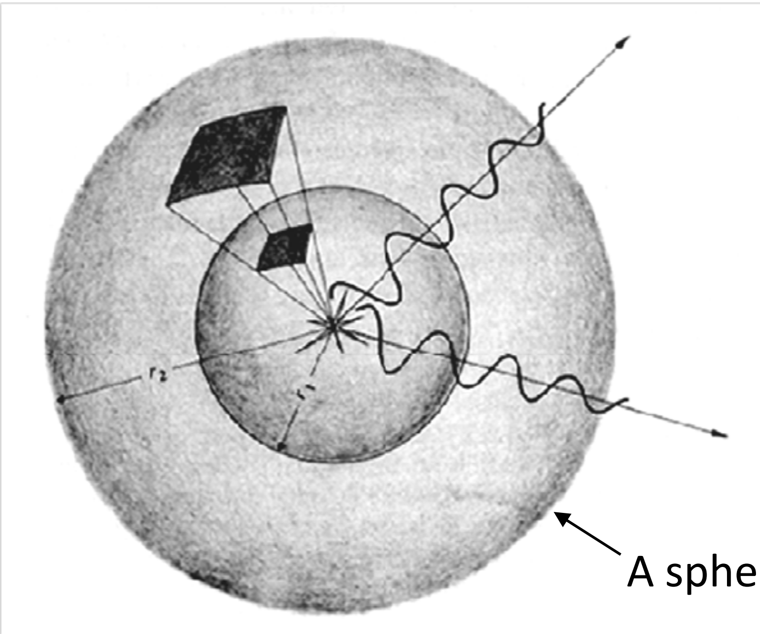
$$\frac{c}{v} = n$$

Phase velocity

*Phase velocity can exceed the speed of light in a dispersive medium where the refractive index n is not necessarily >1.

Spherical waves

A spherical wave is also a solution to Maxwell's equations and is a good model for the light scattered by a molecule.



Note that k and r are **not** vectors here!

$$E(\vec{r},t) \propto \left(E_0 / r\right) \mathrm{Re}\{\exp[i(kr - \omega t)]\}$$

- where k is a scalar, and
- r is the radial magnitude.

A spherical wave has spherical wave-fronts.

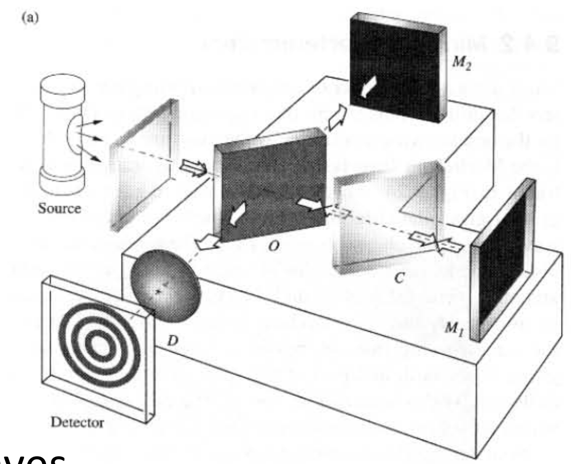
Unlike a plane wave, whose amplitude remains constant as it propagates, a spherical wave weakens. Its irradiance goes as $1/r^2$.

Interference [Hecht 9.1-9.4, 9.7.2; Fowles 3.1-3.1; Notes]

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathrm{Re}\{\mathbf{E}(\mathbf{r})e^{-i\omega t}\} \\ &= \mathrm{Re}\{\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}\} \\ &= \mathrm{Re}\{\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\}\end{aligned}$$

Michelson Interferometer



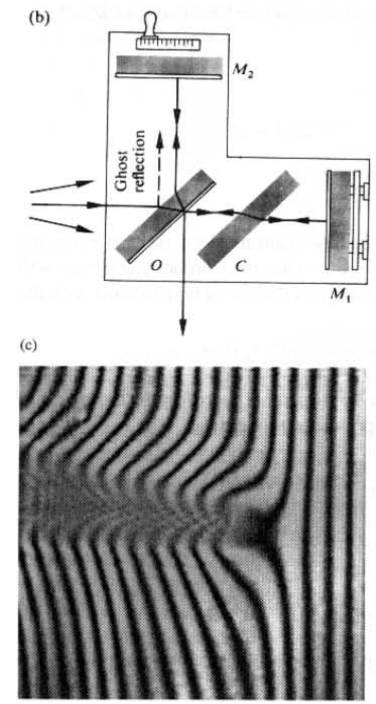


Figure 9.24 The Michelson Interferometer. (*a*) Circular fringes are centered on the lens. (*b*) Top view of the interferometer showing the path of the light. (*c*) A wedge fringe pattern was distorted when the tip of a hot soldering iron was placed in one arm. Observe the interesting perceptual phenomenon whereby the region corresponding to the iron's tip appears faintly yellow. (Photo by E. H.)

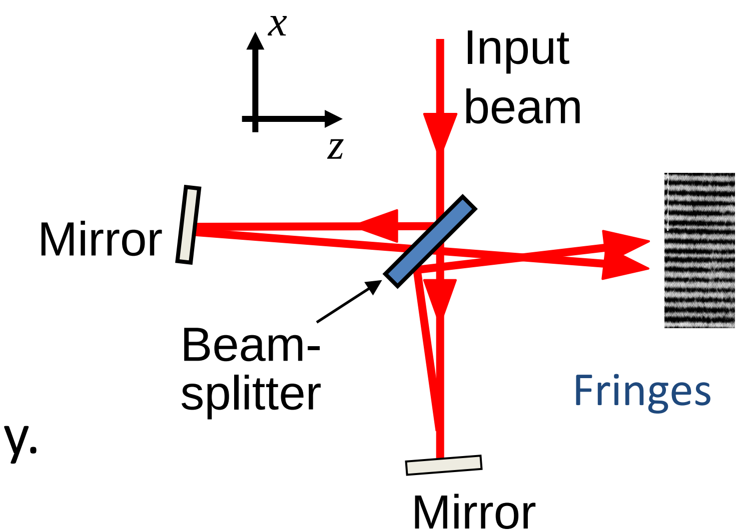
Consider the Optical Path Difference (OPD)
Or simply the superposition of two plane waves

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_1 e^{i\mathbf{k}_1\cdot\mathbf{r}_1} + \mathbf{E}_2 e^{i\mathbf{k}_2\cdot\mathbf{r}_2}$$

$$I = |\mathbf{E}(\mathbf{r})|^2 = \mathbf{E}\times\mathbf{E}^*$$

Key words/Topics:
Michelson Interferometer, Dielectric thin film, Anti-reflection coating, Fringes of equal thickness, Newton rings.

The Michelson Interferometer and Spatial Fringes



- Suppose we misalign the mirrors
- so the beams cross at an angle
- when they recombine at the beam
- splitter. And we won't scan the delay.

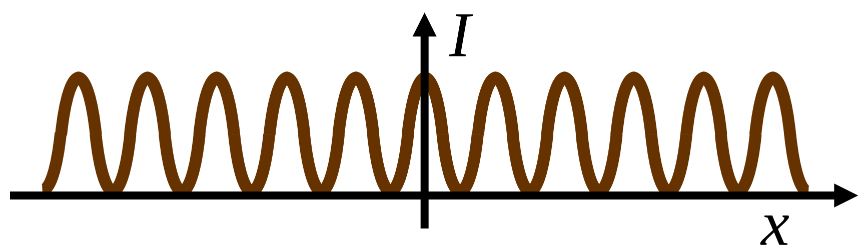
- If the input beam is a plane wave, the cross term becomes:

$$\mathrm{Re}\left\{E_0 \exp\left[i(\cancel{\omega t - kz\cos\theta} - kx\sin\theta\right] E_0^* \exp\left[-i(\cancel{\omega t - kz\cos\theta} + kx\sin\theta\right]\right\}$$

$$\propto \mathrm{Re}\left\{\exp\left[-2ikx\sin\theta\right]\right\}$$

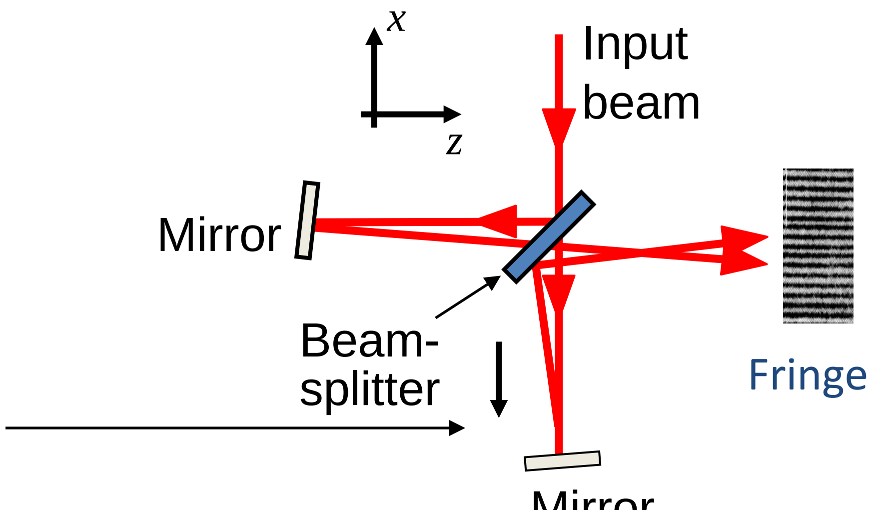
$$\propto \cos(2kx\sin\theta)$$

Fringes (in position)



Crossing beams maps delay onto position.

•Suppose we change one arm's path length.

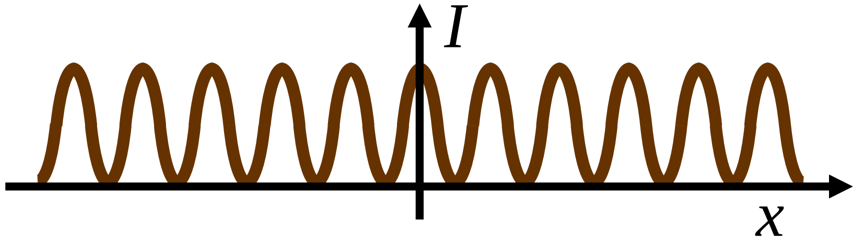


$$\mathrm{Re}\left\{E_0 \exp\left[i(\omega t - kz\cos\theta - kx\sin\theta + 2kd\right] E_0^* \exp\left[-i(\omega t - kz\cos\theta + kx\sin\theta\right]\right\}$$

$$\propto \mathrm{Re}\left\{\exp\left[-2ikx\sin\theta + 2kd\right]\right\}$$

$$\propto \cos(2kx\sin\theta + 2kd)$$

Fringes (in position)

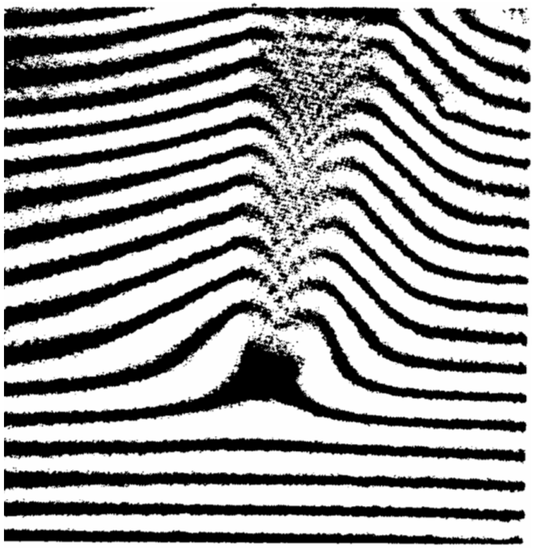


The fringes will shift in phase by $2kd$.

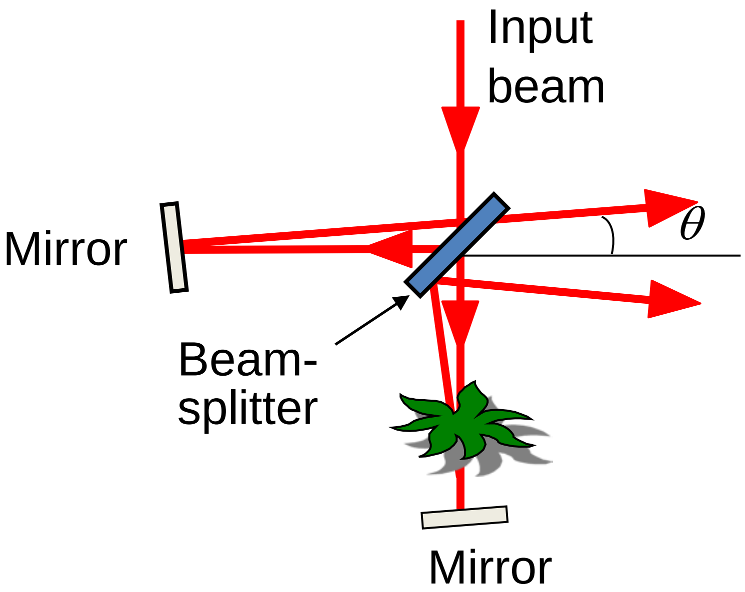
The Unbalanced Michelson Interferometer can sensitively measure phase vs. position.

See HW#8 Problem #1

Spatial fringes distorted by a soldering iron tip in one path

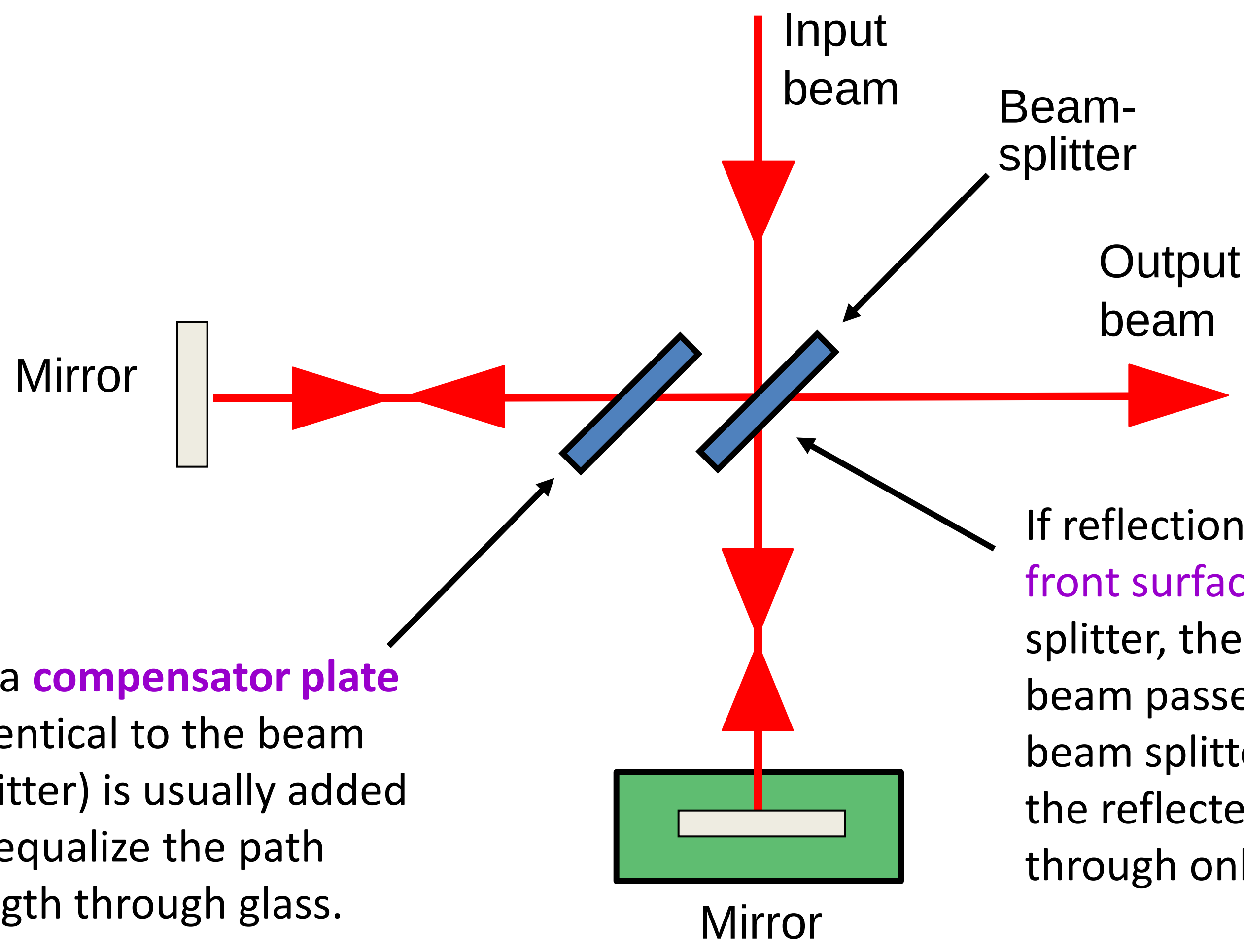


Placing an object in one arm of a misaligned Michelson interferometer will distort the spatial fringes.



- Phase variations of a small fraction of a wavelength can be measured.

Michelson interferometers: the compensator plate



So a **compensator plate** (identical to the beam splitter) is usually added to equalize the path length through glass.

If reflection occurs off the front surface of beam splitter, the transmitted beam passes through beam splitter three times; the reflected beam passes through only once.

Interference Fringes and Newton Rings

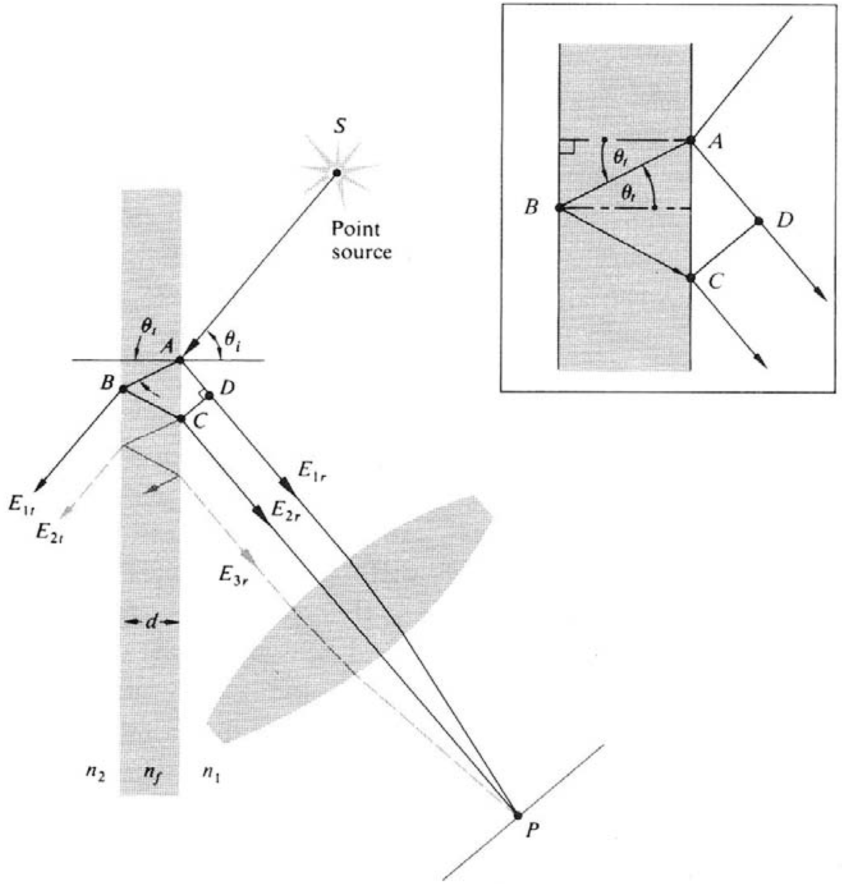
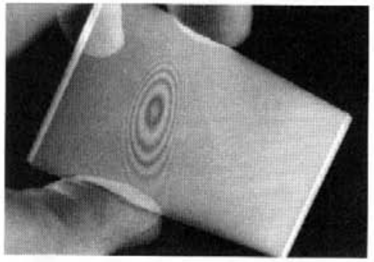


Figure 9.17 Fringes of equal inclination.



Newton's rings with two microscope slides. The thin film of air between the slides creates the interference pattern. (Photo by E. H.)

Newton's Rings

From the figure, if $R \gg d$, then

$$x^2 R^2-(R-d)^2 \Rightarrow x^2 \approx 2Rd$$

The interference maximum will occur if

$$2n_f d_m=(m+\frac{1}{2})\lambda_0$$

Thus, the radius of the bring rings are

$$x_m=\sqrt{(m+\frac{1}{2})\lambda_f R}$$

Similarly, the radius of dark rings are

$$x_m=\sqrt{m\lambda_f R}$$

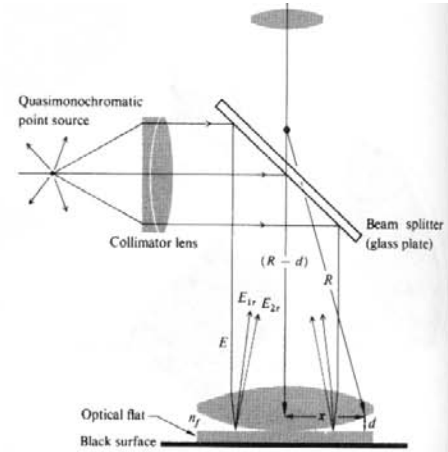
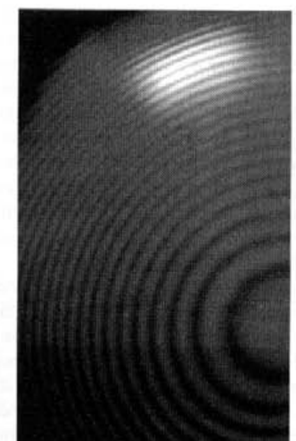


Figure 9.23 A standard setup to observe Newton's rings



Interference from the thin air film between a convex lens and the flat sheet of glass it rests on. The illumination was quasimonochromatic. These fringes were first studied in depth by Newton and are known as Newton's rings. (Photo by E.H.)

Phase shift on reflection at an interface

Near-normal incidence

π phase shift if $n_i < n_t$

0 (or 2π phase shift) if $n_i > n_t$

$$r_\perp = \left(\frac{E_{0r}}{E_{0i}}\right)_\perp = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$t_\perp = \left(\frac{E_{0t}}{E_{0i}}\right)_\perp = \frac{2n_i\cos\theta_i}{n_i\cos\theta_i + n_t\cos\theta_t}$$

$$r_\parallel = \left(\frac{E_{0r}}{E_{0i}}\right)_\parallel = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_t\cos\theta_i + n_i\cos\theta_t}$$

$$t_\parallel = \left(\frac{E_{0t}}{E_{0i}}\right)_\parallel = \frac{2n_i\cos\theta_i}{n_t\cos\theta_i + n_i\cos\theta_t}$$

Note: independent of polarization

$\theta_i = 0$ and $\theta_t = 0$

$$r_\perp = r_\parallel = \frac{n_t - n_i}{n_t + n_i}$$

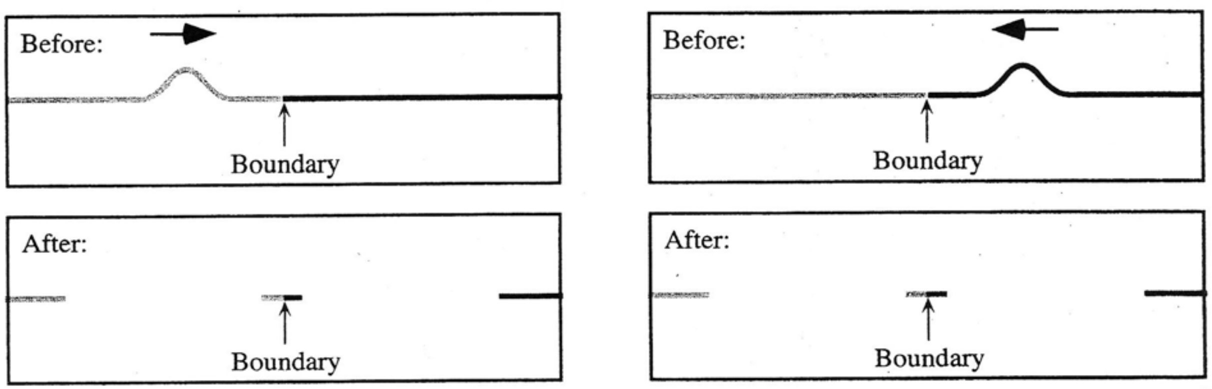
$$t_\perp = t_\parallel = \frac{2n_i}{n_t + n_i}$$

$$R_\perp = R_\parallel = \left(\frac{n_t - n_i}{n_t + n_i}\right)^2$$

$$T_\perp = T_\parallel = \frac{4n_t n_i}{(n_t + n_i)^2}$$

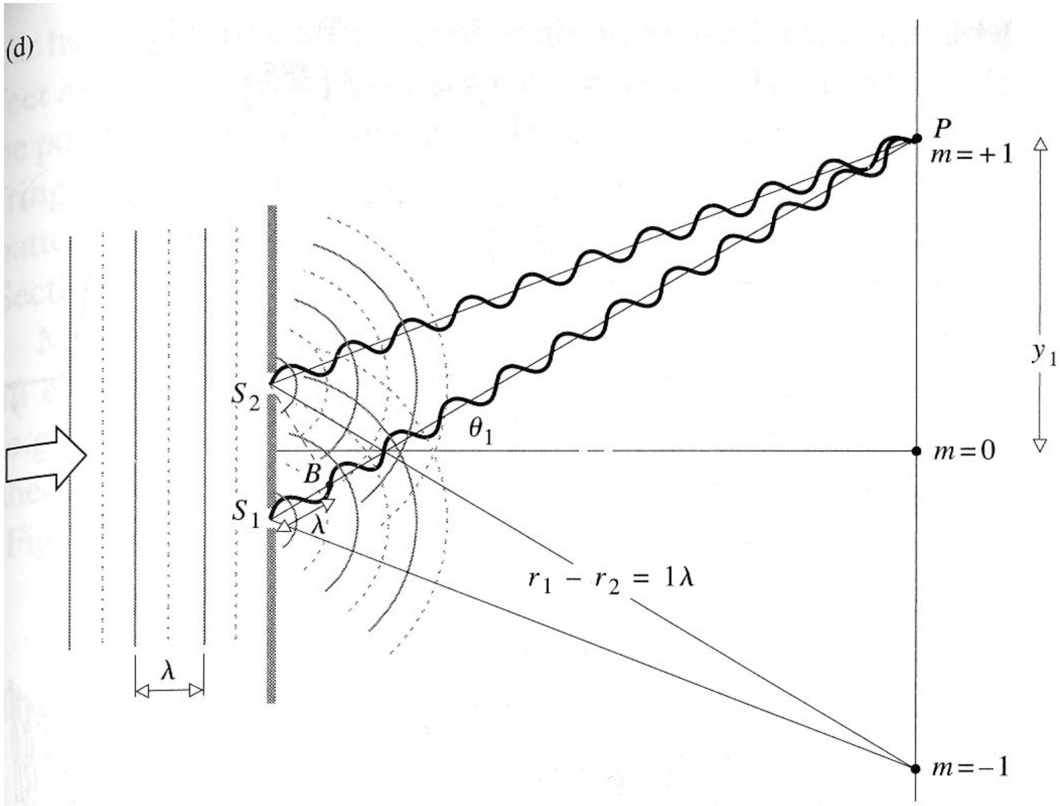
I. Transmission and reflection at a boundary

The sketches below show a pulse approaching a boundary between two springs. In one case, the pulse approaches the boundary from the left; in the other, from the right. The springs are the same in both cases, and the linear mass density is greater for the spring on the right than for the spring on the left.



Complete the sketches to show the shape of the springs a short time after the trailing edge of the pulse shown has reached the boundary. Be sure to show correctly (1) the relative widths of the pulses and (2) which side of the spring each pulse is on. (Ignore relative amplitudes.)

Young's double slit interference experiment



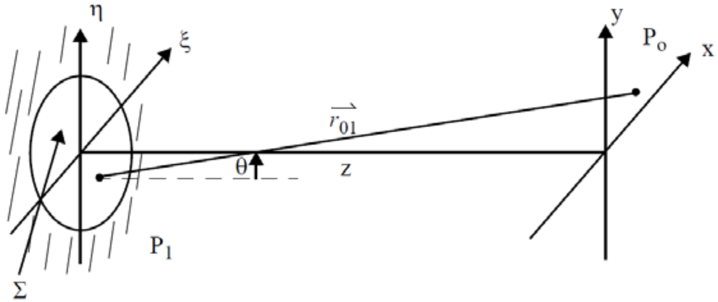
order m maxima occur at:

$$m\lambda \approx a \sin\theta_m \approx a\frac{y_m}{s}$$

Diffraction

Fresnel approximation

Huygens-Fresnel integral in rectangular coordinates:



$$\boxed{r_{01} = [z^2 + (x-\xi)^2 + (y-\eta)^2]^{1/2}}$$

The Fresnel approximation involves setting: $r_{01} \cong z$ in the denominator, and

$$r_{01} \cong z\left[1 + \frac{1}{2}\frac{(x-\xi)^2}{z} + \frac{1}{2}\frac{(y-\eta)^2}{z}\right] \quad \text{in exponent}$$

This is equivalent to the paraxial approximation in ray optics.

$$U(x, y) = \frac{\exp(jkz)}{j\lambda z}\int\int_{-\infty}^{\infty} d\xi d\eta U(\xi, \eta)\exp\left\{\frac{jk}{2z}[(x-\xi)^2 + (y-\eta)^2]\right\} \qquad \text{(A)}$$

Let's examine the validity of the Fresnel approximation in the Fresnel integral. The next higher order term in exponent must be small compared to 1. So the valid range of the Fresnel approximation is:

$$z^3 \gg \frac{\pi}{4\lambda}[(x-\xi)^2 + (y-\eta)^2]^2_{max}$$

For field sizes of 1 cm, $\lambda = 0.5\mu m$, we find $z \gg 25$ cm.

Actually we should look at the effect on the total integral. Upon closer analysis, it is found that the Fresnel approximation holds for a much closer z. This is referred to as the "near-field region".

Farther out in z, we can approximate the quadratic phase as flat

$$z \gg \frac{k(\xi^2 + \eta^2)_{max}}{2}$$

This region is referred to as the "far-field" or Fraunhofer region.

$$U(x, y) = \frac{e^{jkz}e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z}\underbrace{\int\int d\xi d\eta U(\xi, \eta)\exp\left[-j\frac{2\pi}{\lambda z}(x\xi + y\eta)\right]}_{\mathcal{F}\{U(\xi,\eta)\}\big|_{f_x = \frac{x}{\lambda z}, f_y = \frac{y}{\lambda z}}}$$

Now this is exactly the Fourier transform of the aperture distribution with

$$\boxed{f_x = \frac{x}{\lambda z}} \qquad \boxed{f_y = \frac{y}{\lambda z}}$$

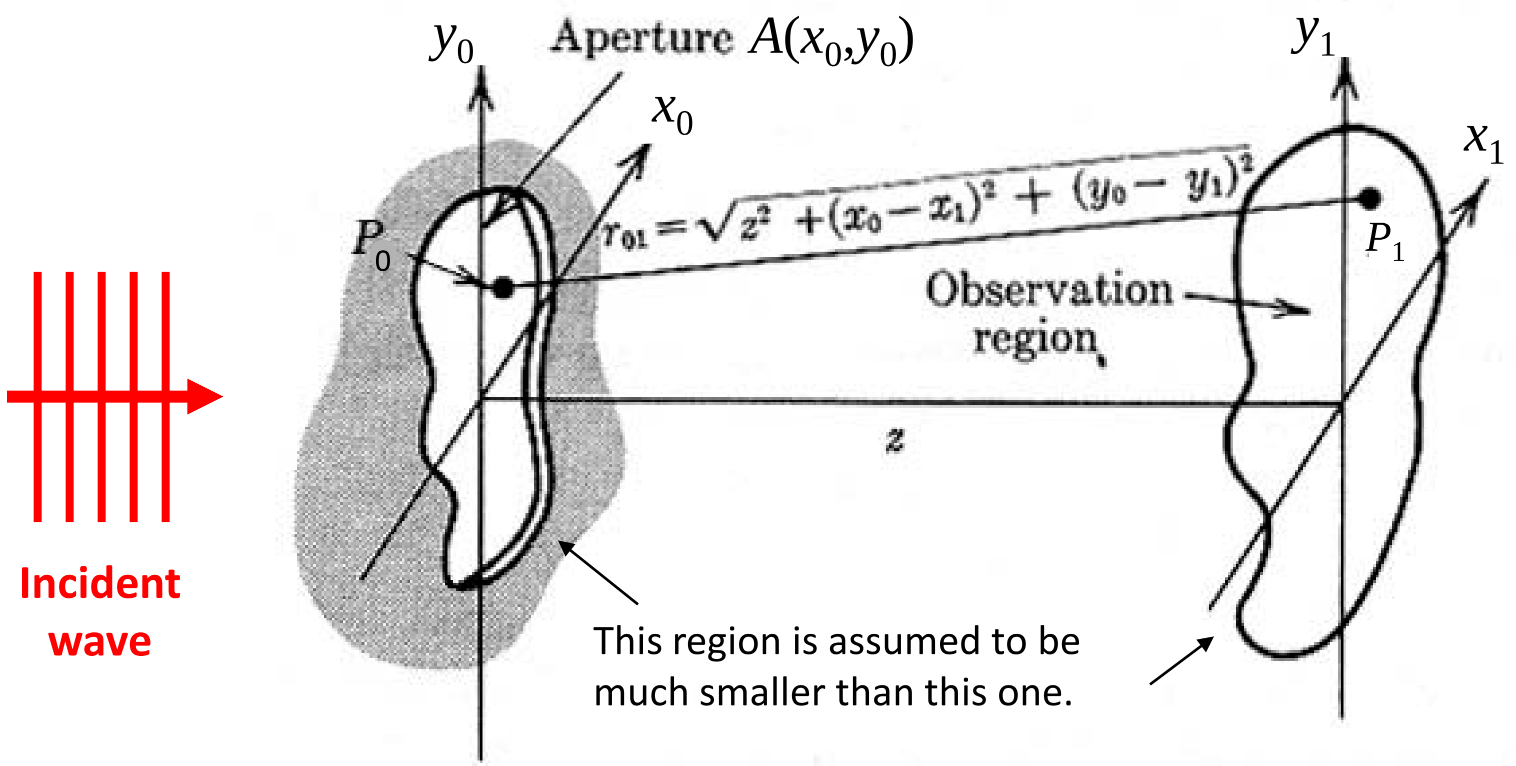
The Fraunhofer region is farther out. For the field size of 1 cm, and $\lambda = 0.5\mu m$, we find the valid range of $z \gg 150$ meters!

Again, examining the full integral, Fraunhofer is actually accurate and usable to much closer distances.

Diffraction Geometry

We wish to find the light electric field after a screen with a hole in it.

This is a very general problem with far-reaching applications.



What is $E(x_1,y_1)$ at a distance z from the plane of the aperture?

Diffraction Solution

The field in the observation plane, $E(x_1,y_1)$, at a distance z from the aperture plane is given by:

$$E(x_1, y_1, z) \;=\; \iint\limits_{A(x_0,y_0)} h(x_1 - x_0, y_1 - y_0, z) E(x_0, y_0)\, dx_0\, dy_0$$

where :

$$h(x_1 - x_0, y_1 - y_0, z) = \frac{1}{i\lambda} \frac{\exp(ikr_{01})}{r_{01}}$$

Spherical wave

and :

$$r_{01} = \sqrt{z^2 + \left(x_0 - x_1\right)^2 + \left(y_0 - y_1\right)^2}$$

A very complicated result! And we cannot approximate r_{01} in the exp by z because it gets multiplied by k, which is big, so relatively small changes in r_{01} can make a big difference!

Fraunhofer Diffraction: The Far Field

We can approximate r_{01} in the denominator by z, and if D is the size of the aperture, $D^2 \geq x_0^{\,2} + y_0^{\,2}$, so when $k\,D^2/\,2z << 1$, the quadratic terms << 1, so we can neglect them:

$$r_{01} = \sqrt{z^2 + \left(x_0 - x_1\right)^2 + \left(y_0 - y_1\right)^2} \approx z\left[1 + \left(x_0 - x_1\right)^2 / 2z^2 + \left(y_0 - y_1\right)^2 / 2z^2\right]$$

$$kr_{01} \approx kz + k\left(x_0^2 - 2x_0x_1 + x_1^2\right)/2z + k\left(y_0^2 - 2y_0y_1 + y_1^2\right)/2z$$

Small, so neglect these terms.

Independent of x_0 and y_0, so factor these out.

$$E\left(x_1, y_1\right) = \frac{\exp(ikz)}{i\lambda z}\exp\left[ik\frac{x_1^2 + y_1^2}{2z}\right]\iint_{A(x_0,y_0)} \exp\left\{-\frac{ik}{z}\left(x_0x_1 + y_0y_1\right)\right\} E\left(x_0, y_0\right) dx_0\, dy_0$$

This condition means going a distance away: $z >> kD^2/2 = \pi D^2/\lambda$

If $D = 1$ mm and $\lambda = 1$ micron, then $z >> 3$ m.

Fraunhofer Diffraction

We'll neglect the phase factors, and we'll explicitly write the aperture function in the integral:

$$\boxed{E(x_1, y_1) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{ik}{z}\left(x_0 x_1 + y_0 y_1\right)\right\} A(x_0, y_0)\, E(x_0, y_0)\, dx_0\, dy_0}$$

$E(x_0,y_0)$ = constant if a plane wave

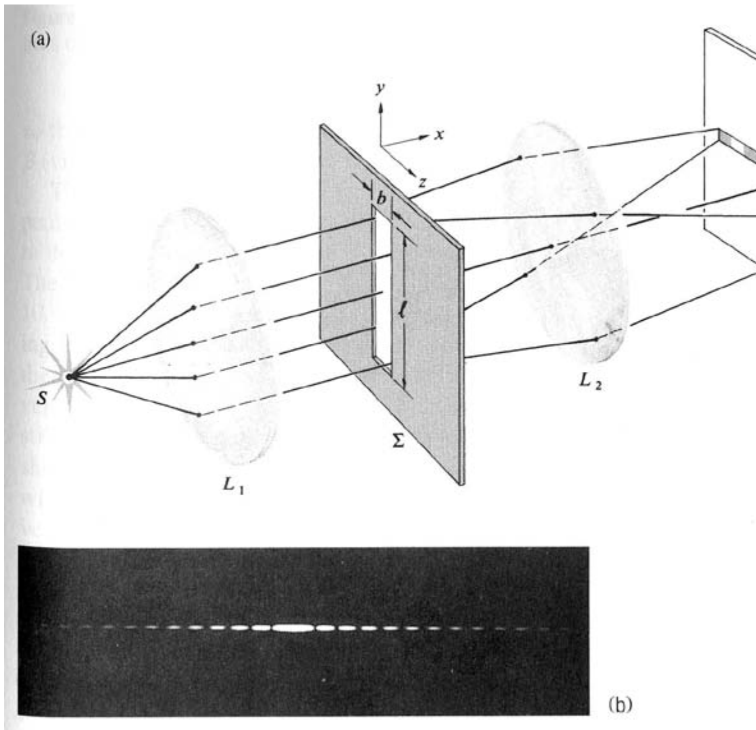
This is just a Fourier Transform!

Interestingly, it's a Fourier Transform from position, x_0, to another position variable, x_1 (in another plane). Usually, the Fourier "conjugate variables" have reciprocal units (e.g., t & ω, or x & k). The conjugate variables here are really x_0 and $k_x = kx_1/z$, which have reciprocal units.

So the far-field light field is the Fourier Transform of the apertured field!

Diffraction: single, double, multiple slits

Study Guide: Hecht Ch. 10.2.1-10.2.6 (detailed lengthy discussions), Fowles Ch. 5 (short but clear presentation), or Lecture Notes



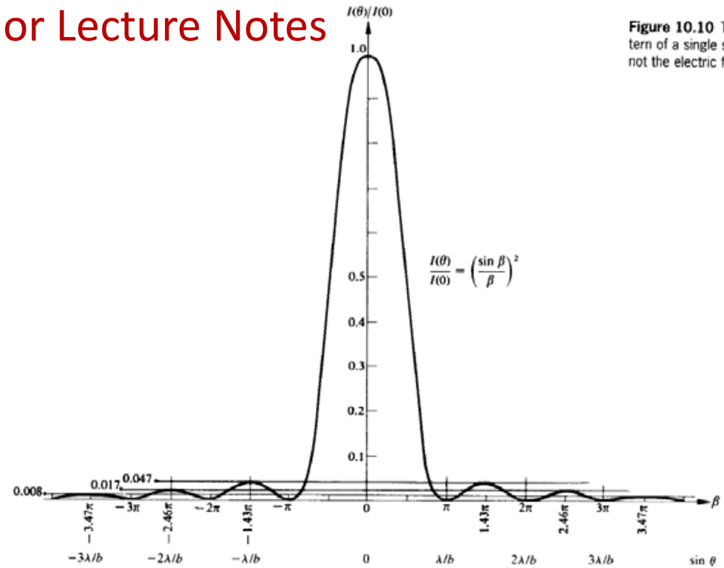
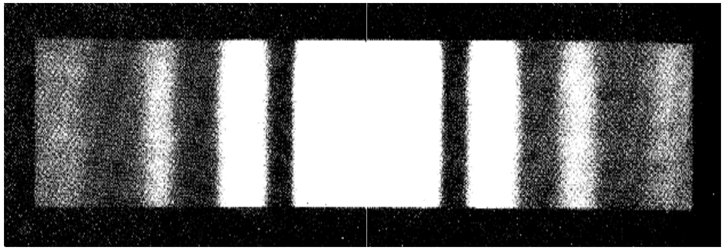


Figure 10.10 The Fraunhofer diffraction pattern of a single slit. This is the irradiance (and not the electric field) distribution

Single Slit ($\Delta x \ll \Delta y \Rightarrow \beta_x \ll \beta_y$)

$\mathrm{sinc}(\beta_y)$ changes much faster than $\mathrm{sinc}(\beta_x)$



$$I(\beta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2$$

$$\beta = \frac{kb}{2}\sin\theta = \pi\frac{b}{\lambda}\sin\theta$$

Java applet – Single Slit Diffraction
http://www.walter-fendt.de/ph14e/singleslit.htm

Diffraction: Double and Multiple Slits

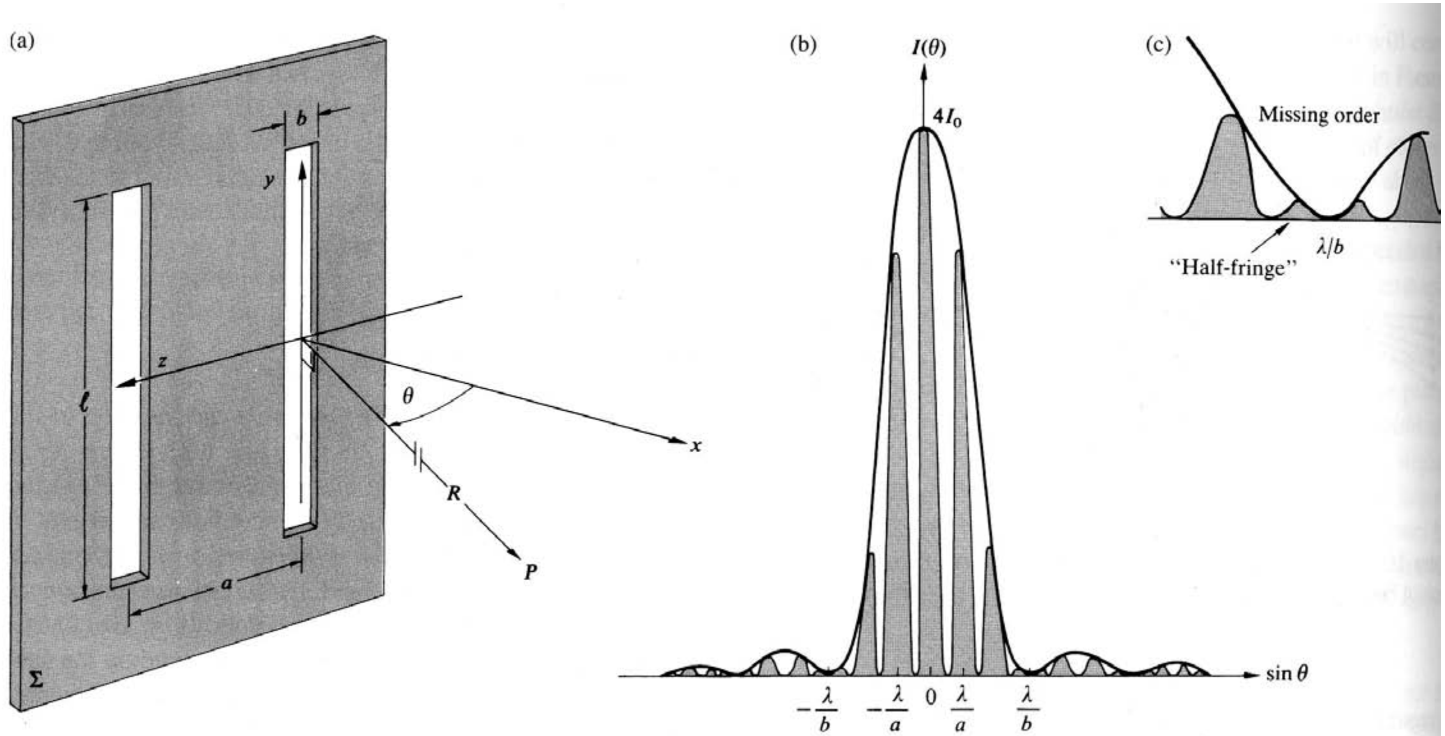
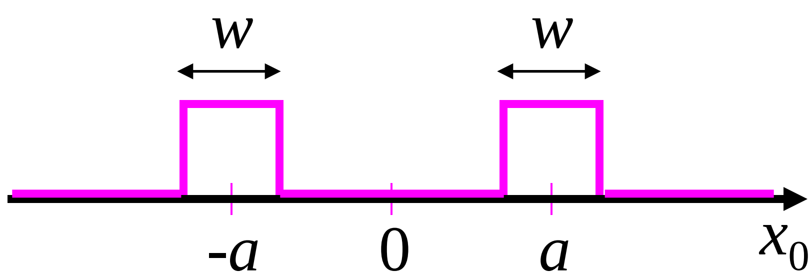


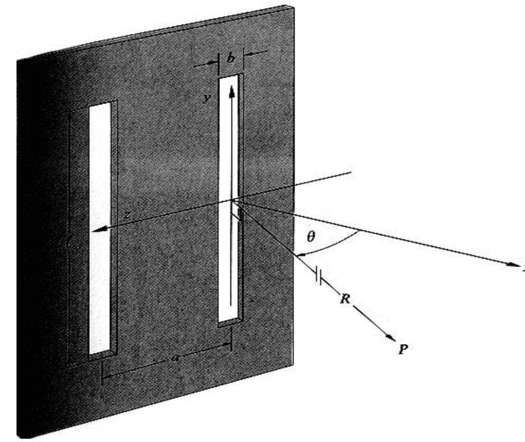
Figure 10.13 (*a*) Double-slit geometry. Point *P* on σ is essentially infinitely far away. (*b*) A double-slit pattern ($a = 3b$).

$$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\gamma}{N\sin\gamma}\right)^2 \quad \beta = \frac{1}{2}kb\sin\theta; \quad \gamma = \frac{1}{2}ka\sin\theta$$

See also
http://demonstrations.wolfram.com/MultipleSlitDiffractionPattern/ and
http://wyant.optics.arizona.edu/multipleSlits/multipleSlits.htm

Fraunhofer diffraction from two slits (Fourier Transform)



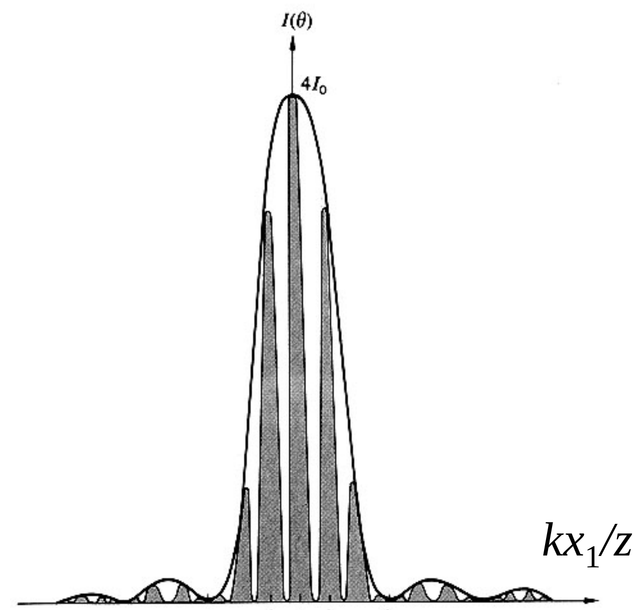


$$A(x_0) = \mathrm{rect}[(x_0+a)/w] + \mathrm{rect}[(x_0-a)/w]$$

$$E(x_1) \propto \mathscr{F}\{A(x_0)\}$$

$$\propto \mathrm{sinc}[w(kx_1/z)/2]\exp[+ia(kx_1/z)]+$$
$$\mathrm{sinc}[w(kx_1/z)/2]\exp[-ia(kx_1/z)]$$

$$\boxed{E(x_1) \propto \mathrm{sinc}(wkx_1/2z)\cos(akx_1/z)}$$

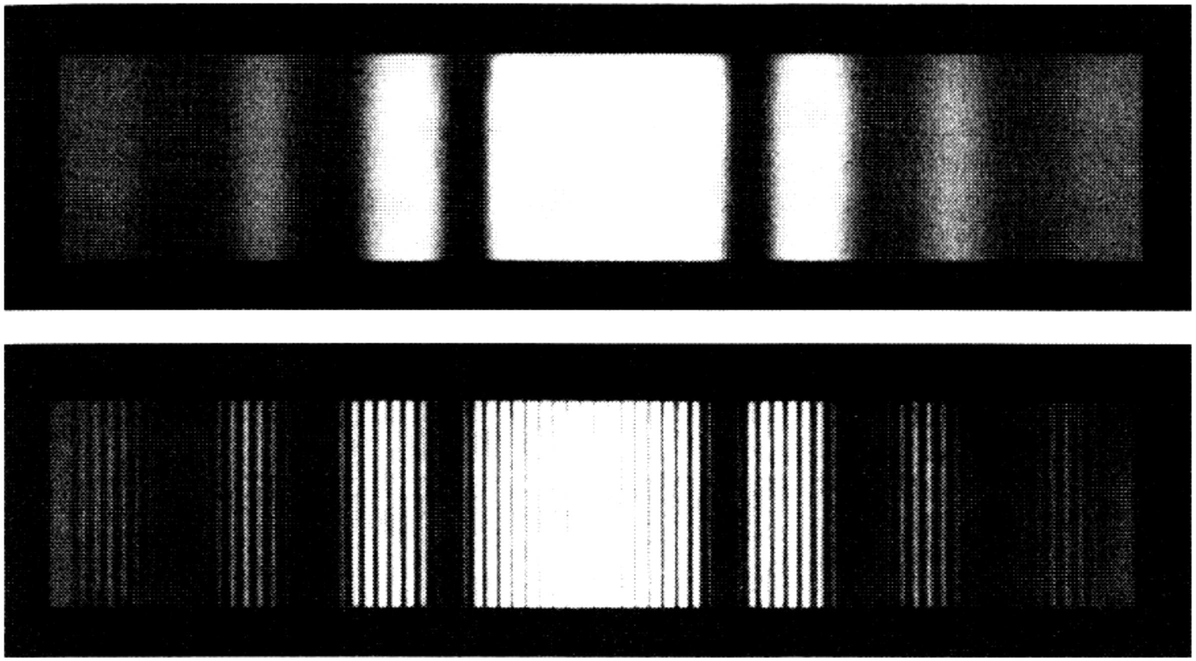


Diffraction from one- and two-slit screens

Fraunhofer diffraction patterns

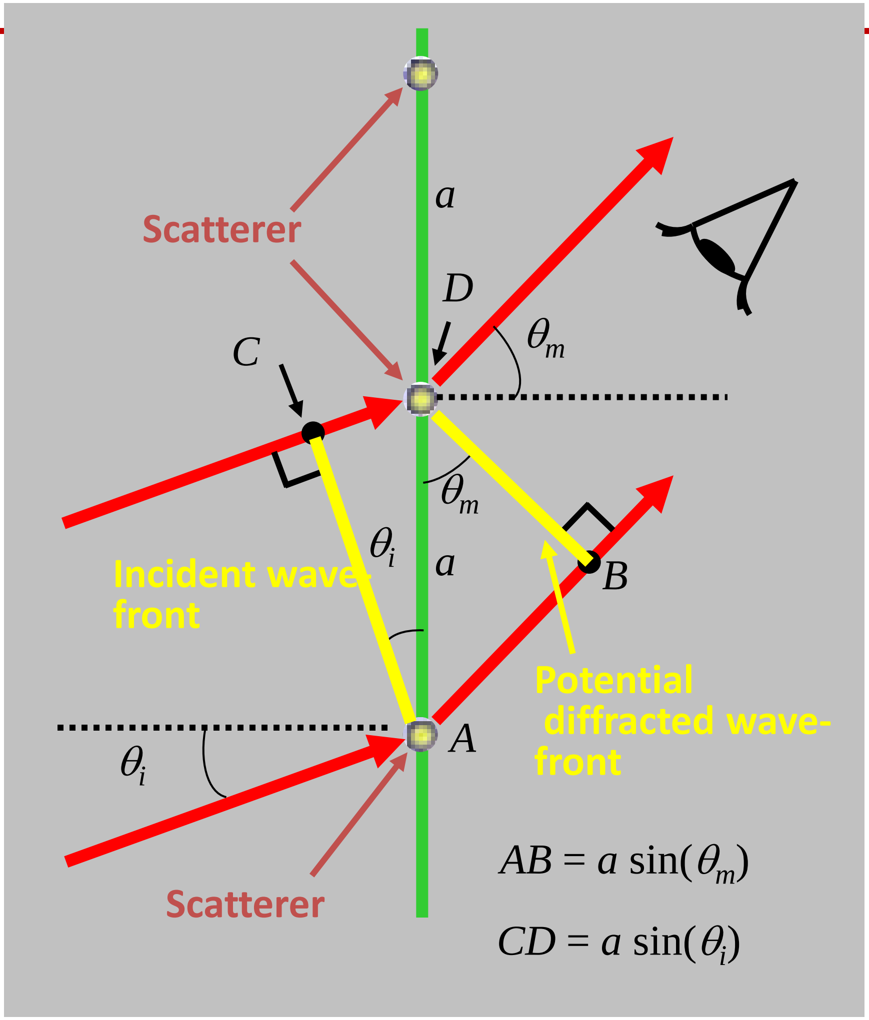
One slit

Two slits



Diffraction Gratings

•Scattering ideas explain what happens when light impinges on a periodic array of grooves. Constructive interference occurs if the delay between adjacent beamlets is an integral number, m, of wavelengths.



Path difference: $AB - CD = m\lambda$

$$\boxed{a\left[\sin(\theta_m) - \sin(\theta_i)\right] = m\lambda}$$

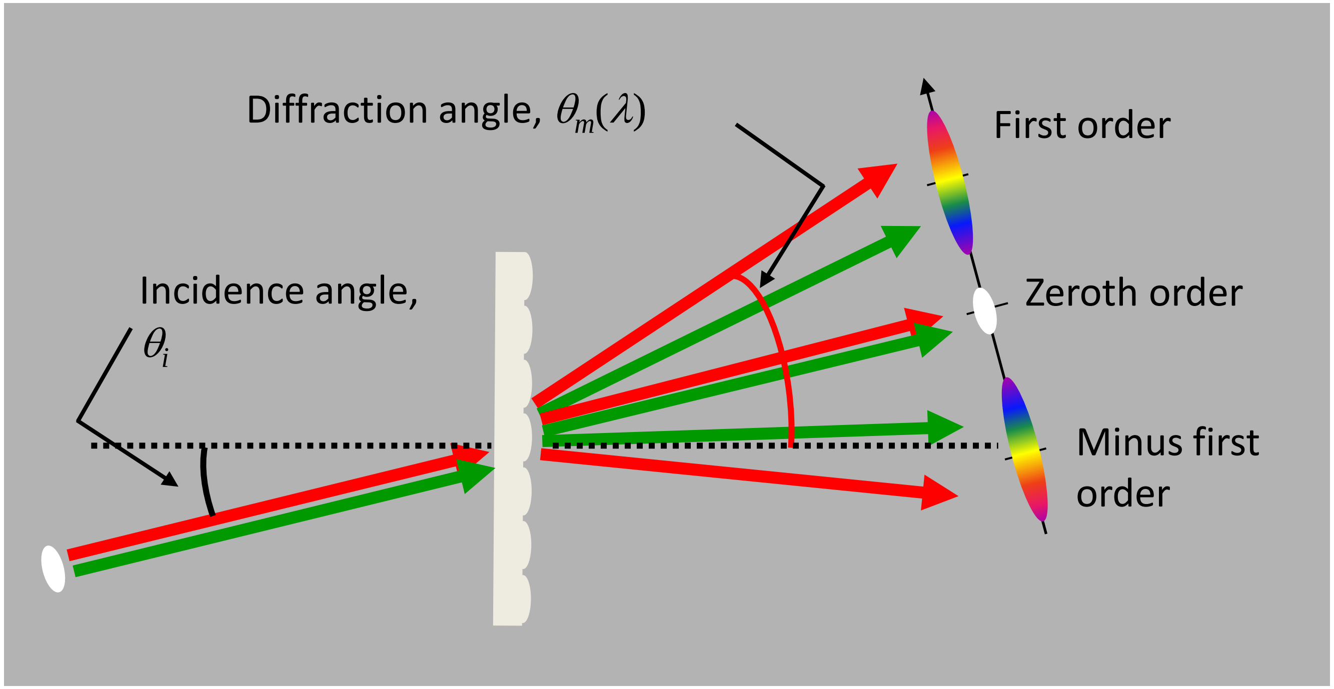
where m is any integer.

A grating has solutions of zero, one, or many values of m, or **orders**.

Remember that m and θ_m can be negative, too.

Diffraction orders

Because the diffraction angle depends on λ, different wavelengths are separated in the nonzero orders.



No wavelength dependence occurs in zero order.

The longer the wavelength, the larger its deflection in each nonzero order.

The Diffraction Grating

Hecht 10.2.8 or Fowles Ch. 5 p.123 (handout)

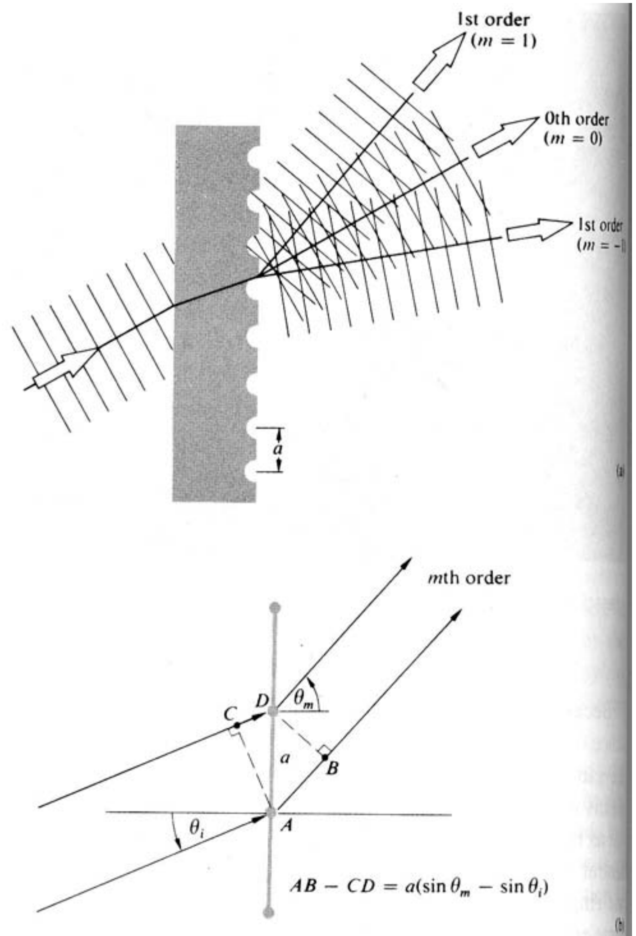


Figure 10.28 A transmission grating.

Grating Equation

(Optical Path Difference OPD= m λ)

$$a\left(\sin\theta_m - \sin\theta_i\right) = m\lambda$$

$$a\sin\theta_m = m\lambda$$ **Normal incidence θ_i =0**

The chromatic/spectral resolving power of a grating

$$R \equiv \frac{\lambda}{\Delta\lambda} = mN$$

m is the order number, and
N is the total number of gratings.

Uniform Rectangular Aperture

Uniform Rectangular Aperture

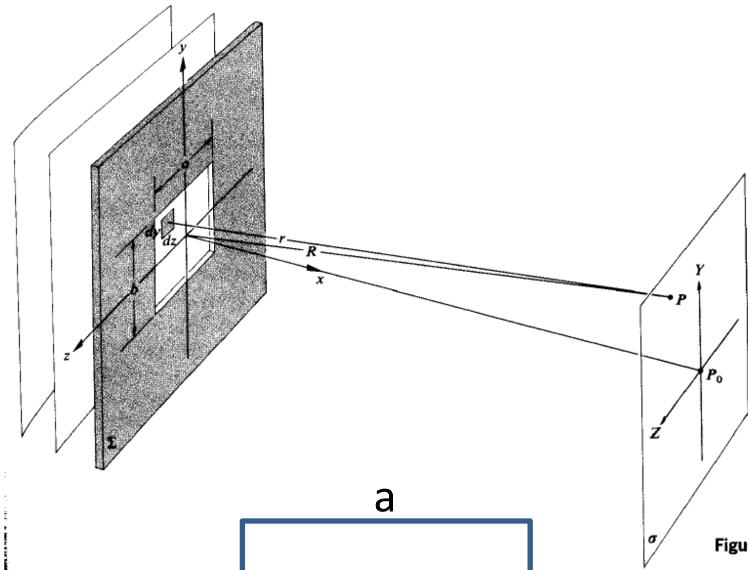
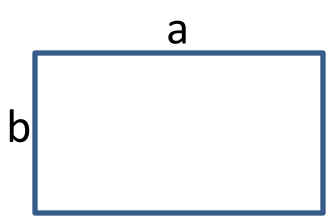
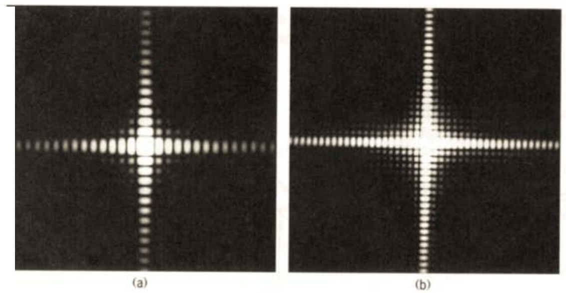


Figure 10.19 A rectangular aperture.





(a) Fraunhofer pattern of a square aperture. (b) The same pattern further exposed to bring out some of the faint terms. (Photos by E. H.)

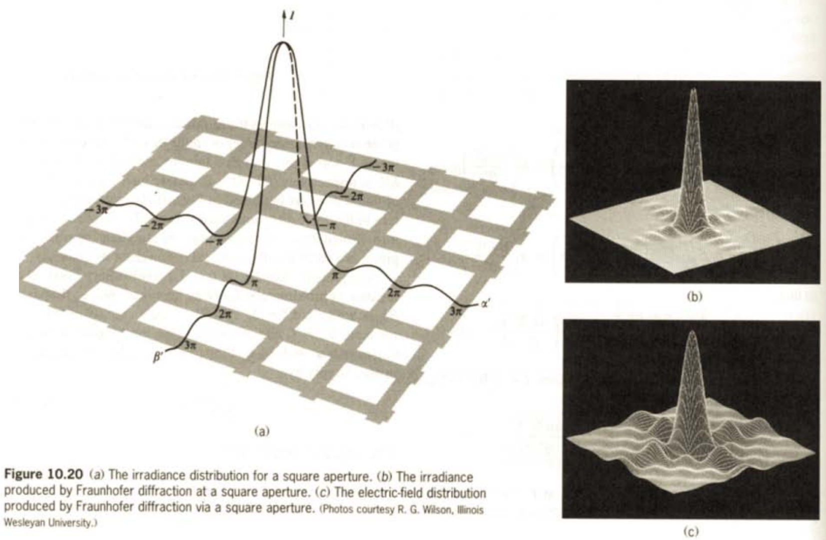
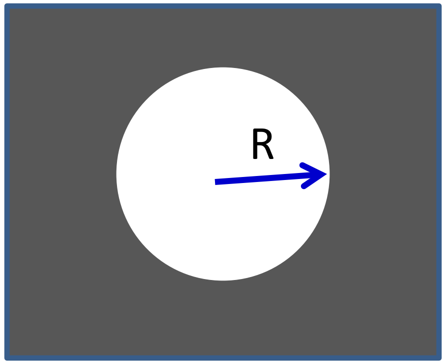
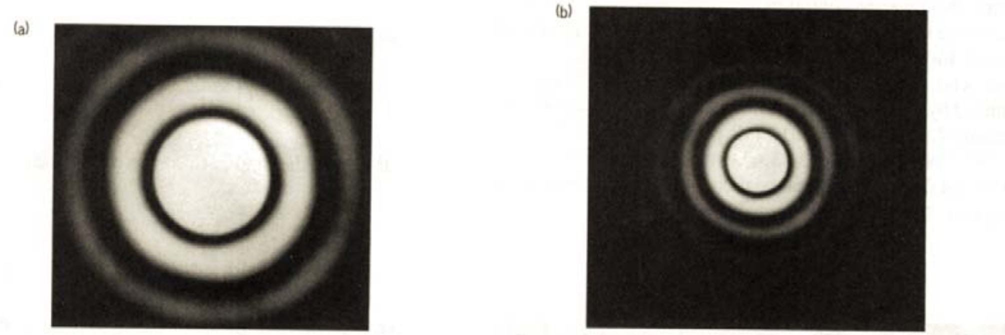


Figure 10.20 (a) The irradiance distribution for a square aperture. (b) The irradiance produced by Fraunhofer diffraction at a square aperture. (c) The electric-field distribution produced by Fraunhofer diffraction via a square aperture. (Photos courtesy R. G. Wilson, Illinois Wesleyan University.)

$$I(\theta) = I(0)\left(\frac{\sin\alpha}{\alpha}\right)^2\left(\frac{\sin\beta}{\beta}\right)^2 \quad \alpha = \frac{1}{2}ka\sin\theta; \quad \beta = \frac{1}{2}kb\sin\theta$$

Uniform Circular Aperture





Airy rings using (a) a 0.5-mm hole diameter and (b) a 1.0-mm hole diameter. (Photo by E. H.)

$$I(\theta) = I(0)\left(\frac{2J_1(\rho)}{\rho}\right)^2$$

$$\rho = kR\sin\theta; \quad k = \frac{2\pi}{\lambda}$$

A circular aperture yields a diffracted "Airy Pattern," which involves a Bessel function.

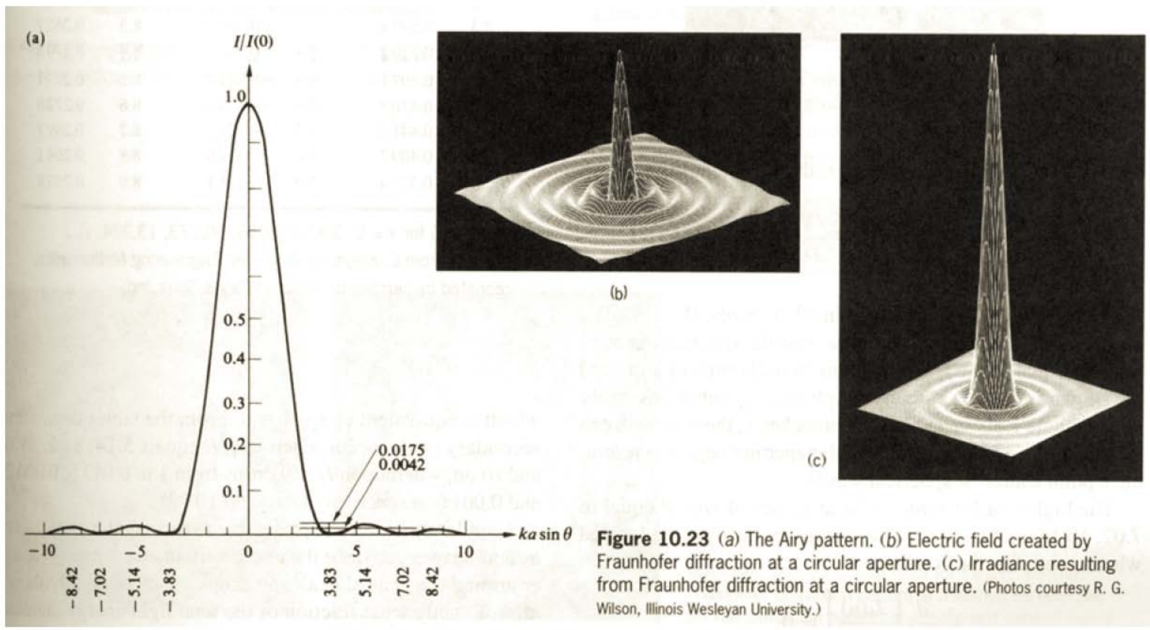
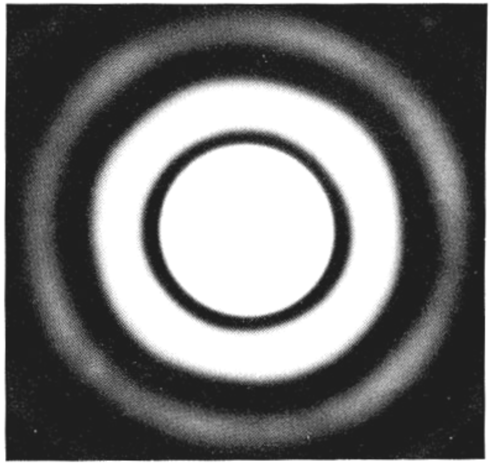


Figure 10.23 (*a*) The Airy pattern. (*b*) Electric field created by Fraunhofer diffraction at a circular aperture. (*c*) Irradiance resulting from Fraunhofer diffraction at a circular aperture. (Photos courtesy R. G. Wilson, Illinois Wesleyan University.)

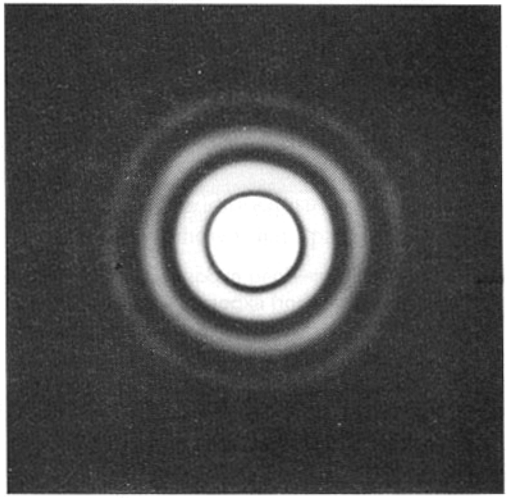
Diffraction from small and large circular apertures

Far-field intensity pattern from a small aperture



Recall the Scale Theorem! This is the Uncertainty Principle for diffraction.

Far-field intensity pattern from a large aperture



Wave optics of a lens

We have previously seen that light passing through a lens experiences a phase delay given by:

$$\varphi(x,y) = \exp\left[-jk(n-1)\left(\frac{x^2+y^2}{2}\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right] \quad \text{(neglecting the constant phase)}$$

The focal length, f is given by:

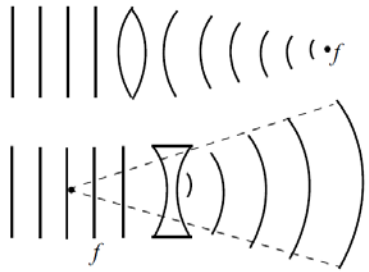
$$\boxed{\frac{1}{f} \equiv (n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} \quad \text{The "lens makers formula"}$$

The transmission function is now:

$$\boxed{\varphi(x,y) = \exp\left[-j\frac{k}{2f}(x^2+y^2)\right]}$$

This is the paraxial approximation to the spherical phase

Note: the incident plane-wave is converted to a spherical wave converging to a point at f behind the lens (f positive) or diverging from the point at f in front of lens (f negative).



Diffraction from the lens pupil

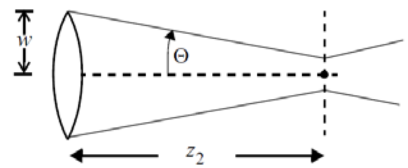
Suppose the lens is illuminated by a plane wave, amplitude A. The lens "pupil function" is $P(x,y)$.

The full effect of the lens is $U_l'(x,y) = \varphi(x,y)P(x,y)$

The focal plane amplitude distribution is a Fourier transform of the lens pupil function $P(x,y)$, multiplied by a quadratic phase term. However, the intensity distribution is exactly

$$I_f(u,v) = \frac{A^2}{\lambda^2 f^2}\left|\mathcal{F}[P(x,y)]\right|^2 \qquad f_x = \frac{u}{\lambda f} \qquad f_y = \frac{v}{\lambda f}$$

Example: a circular lens, with radius w



$$P = \mathrm{circ}\left(\frac{q}{w}\right) \qquad (q^2 = x^2+y^2)$$

$$\text{let} \quad h(r) = \mathcal{F}[P(\lambda z_2 q)] = \mathcal{F}\left[\mathrm{circ}\left(\frac{\lambda z_2 q}{w}\right)\right] \qquad (r^2 = u^2+v^2)$$

$$= \frac{A}{\lambda z_2}\left[2\frac{J_1(2\pi w r/\lambda z_2)}{2\pi w r/\lambda z_2}\right]$$

$$|h(r)|^2 = \frac{A^2}{\lambda^2 z_2^2}\left[2\frac{J_1(2\pi w r/\lambda z_2)}{2\pi w r/\lambda z_2}\right]^2$$

The spot diameter is

$$d = 1.22\frac{\lambda f}{w} = 1.22\frac{\lambda}{\theta}$$

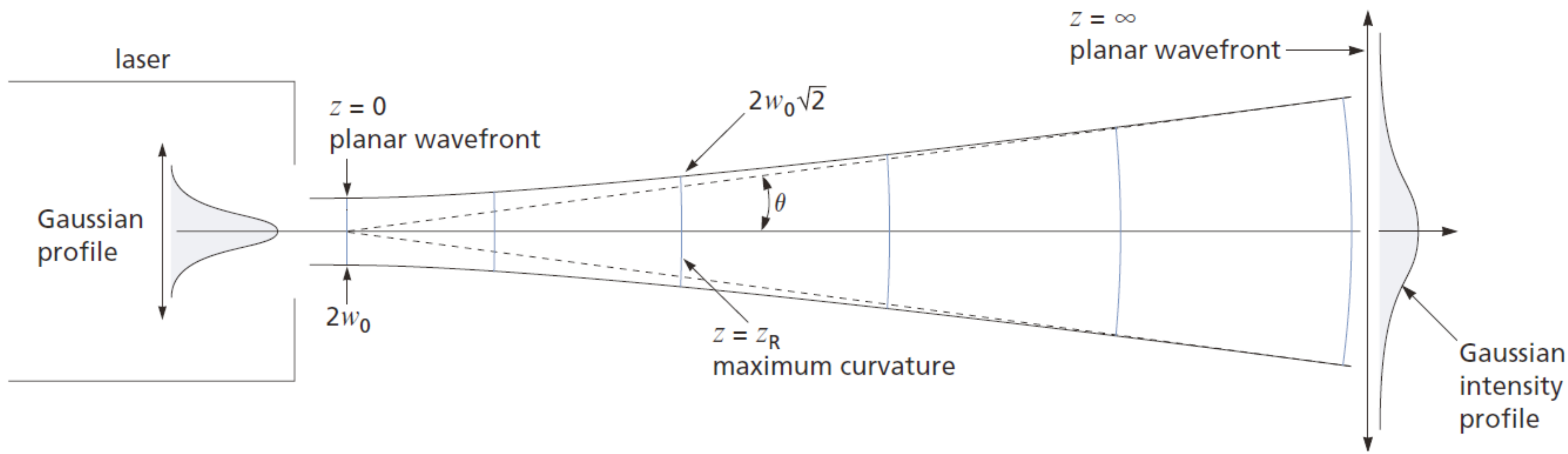
The resolution of the lens as defined by the "Rayleigh" criterion is

$$d/2 = 0.61\lambda/\theta$$

For a small angle θ,

$$d/2 = 0.61\lambda/\sin\theta = 0.61\frac{\lambda}{NA}$$

Gaussian Beam Optics (only eq. (4))



$$I(r) = I_0 e^{-2r^2/w^2} = \frac{2P}{\pi w^2} e^{-2r^2/w^2}$$

$$R(z) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]$$

and

$$w(z) = w_0\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right]^{1/2}$$

$$w(z) = w_o\left[1 + \left(\frac{\lambda z}{\pi w_o^2}\right)^2\right]^{1/2} = w_o\left[1 + \left(\frac{z}{z_R}\right)^2\right]^{1/2} \qquad (2)$$

where we have defined a new parameter, called the Rayleigh range,

$$z_R = \pi w_o^2 / \lambda, \qquad (3)$$

which combines the wavelength and waist radius into a single parameter and completely describes the divergence of the Gaussian beam. Note that the Rayleigh range is the distance from the beam waist to the point at which the beam radius has increased to $\sqrt{2}w_o$. For a 633 nm red He-Ne laser with a waist of 0.4 mm, $z_R \approx 0.8$ m.

When $z >> z_R$, Eq. (2) simplifies to $w = w_o z / z_R$ and the laser beam diverges at a constant angle

$$\theta = \frac{w}{z} = \frac{w_o}{z_R} = \frac{\lambda}{\pi w_o} \qquad (4)$$

Note that the smaller the Rayleigh range, the more rapidly the beam diverges.

Basic Fourier Optics (~30-50 points)

Fourier Transform Notation

There are several ways to denote the Fourier transform of a function.

If the function is labeled by a lower-case letter, such as f,
we can write:

$$f(t) \rightarrow F(\omega)$$

If the function is already labeled by an upper-case letter, such as E, we can write:

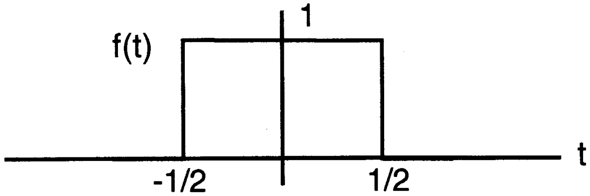
$$E(t) \rightarrow \mathscr{F}\{E(t)\}$$

or:

$$E(t) \rightarrow \tilde{E}(\omega)$$

Sometimes, this symbol is used instead of the arrow: $\supset$

Example: the Fourier Transform of a rectangle function: rect(t)



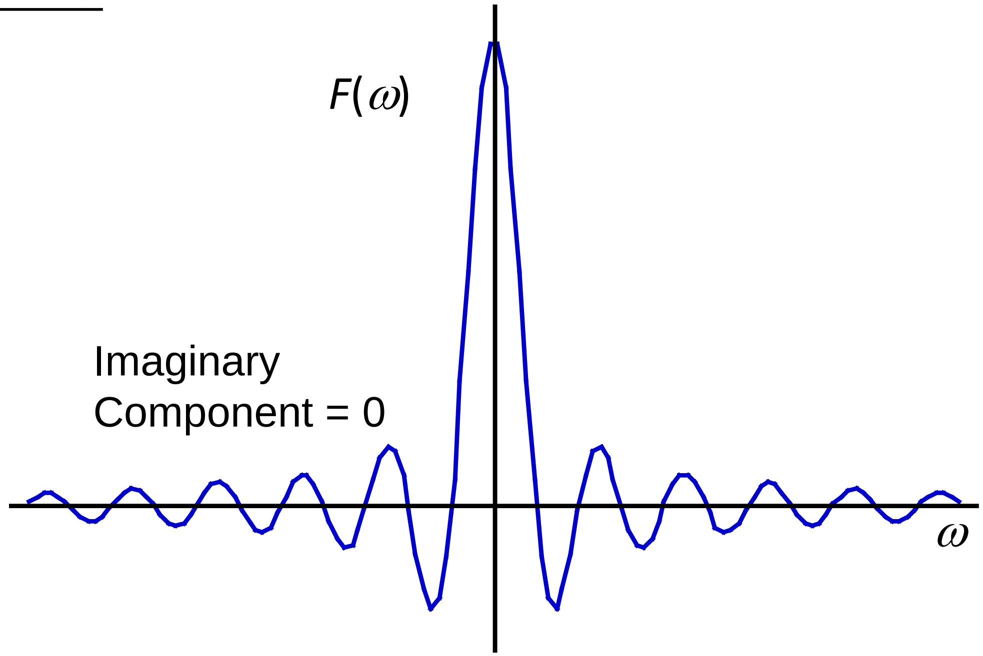
$$F(\omega) = \int_{-1/2}^{1/2} \exp(-i\omega t)\,dt = \frac{1}{-i\omega}\left[\exp(-i\omega t)\right]_{-1/2}^{1/2}$$

$$= \frac{1}{-i\omega}\left[\exp(-i\omega/2) - \exp(i\omega/2)\right]$$

$$= \frac{1}{(\omega/2)} \frac{\exp(i\omega/2) - \exp(-i\omega/2)}{2i}$$

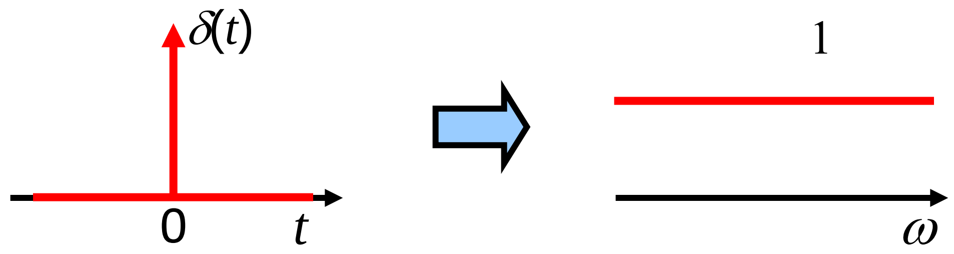
$$= \frac{\sin(\omega/2)}{(\omega/2)}$$

$$\boxed{F(\omega) = \mathrm{sinc}(\omega/2)}$$

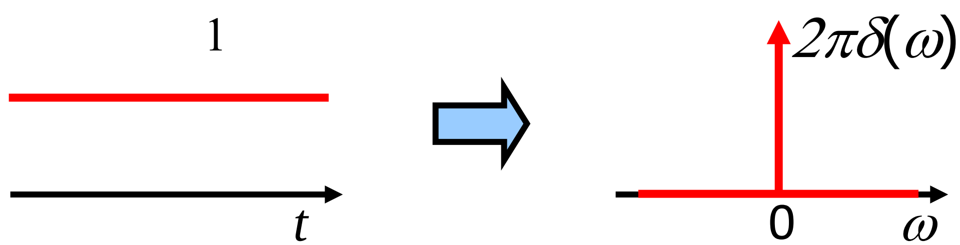


The Fourier Transform of $\delta(t)$ is 1.

$$\int_{-\infty}^{\infty} \delta(t)\exp(-i\omega t)\,dt = \exp(-i\omega[0]) = 1$$



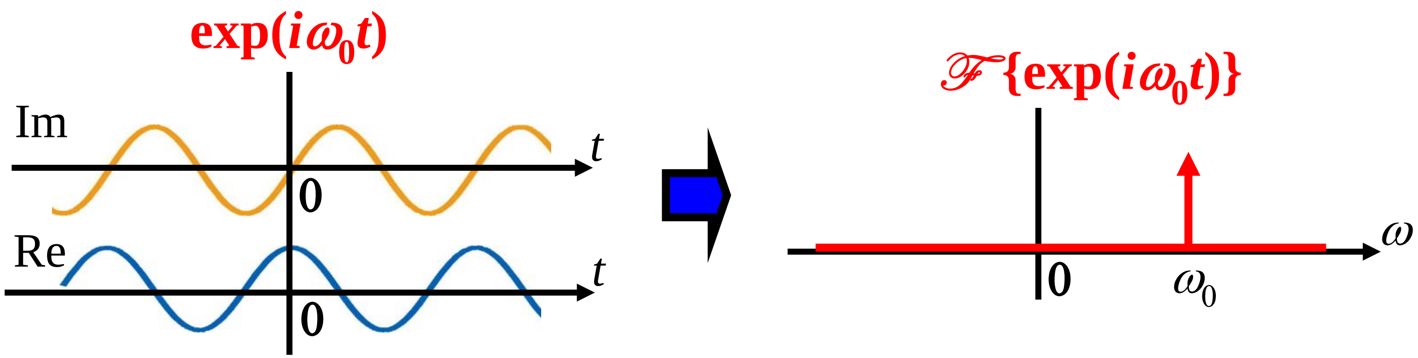
And the Fourier Transform of 1 is $2\pi\delta(\omega)$: $$\int_{-\infty}^{\infty} 1\exp(-i\omega t)\,dt = 2\pi\,\delta(\omega)$$



The Fourier transform of $\exp(i\omega_0 t)$

$$\mathcal{F}\{\exp(i\omega_0 t)\} = \int_{-\infty}^{\infty} \exp(i\omega_0 t)\exp(-i\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-i[\omega-\omega_0]t)\,dt = 2\pi\,\delta(\omega-\omega_0)$$



The function $\exp(i\omega_0 t)$ is the essential component of Fourier analysis.

It is a pure frequency.

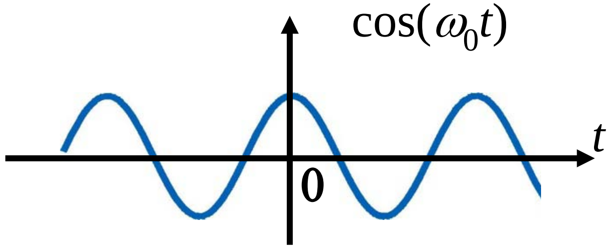
The Fourier transform of $\cos(\omega_0 t)$

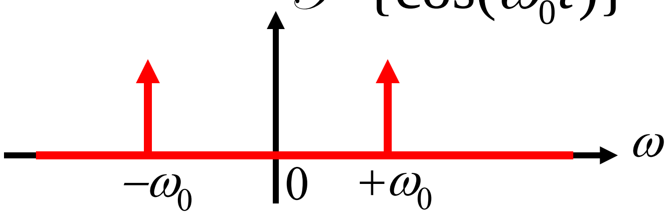
$$\mathscr{F}\{\cos(\omega_0 t)\} = \int_{-\infty}^{\infty} \cos(\omega_0 t) \exp(-i\,\omega t)\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[\exp(i\,\omega_0\, t) + \exp(-i\,\omega_0\, t)\right] \exp(-i\,\omega t)\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega - \omega_0]t)\, dt \quad + \quad \frac{1}{2}\int_{-\infty}^{\infty} \exp(-i[\omega + \omega_0]t)\, dt$$

$$= \pi\,\delta(\omega - \omega_0) \quad + \quad \pi\,\delta(\omega + \omega_0)$$





Scale Theorem

The Fourier transform of a scaled function, $f(at)$:

$$\boxed{\mathscr{F}\{f(at)\} = F(\omega/a)\,/\,|a|}$$

Proof:

$$\mathscr{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\exp(-i\omega t)\,dt$$

Assuming $a > 0$, change variables: $u = at$

$$\mathscr{F}\{f(at)\} = \int_{-\infty}^{\infty} f(u)\exp(-i\omega[u/a])\,du\,/\,a$$

$$= \int_{-\infty}^{\infty} f(u)\exp(-i\,[\omega/a]\,u)\,du\;\;/\;\;a$$

$$= F(\omega/a)\,/\,a$$

If $a < 0$, the limits flip when we change variables, introducing a minus sign, hence the absolute value.

The Scale Theorem in action

The shorter the pulse, the broader the spectrum!

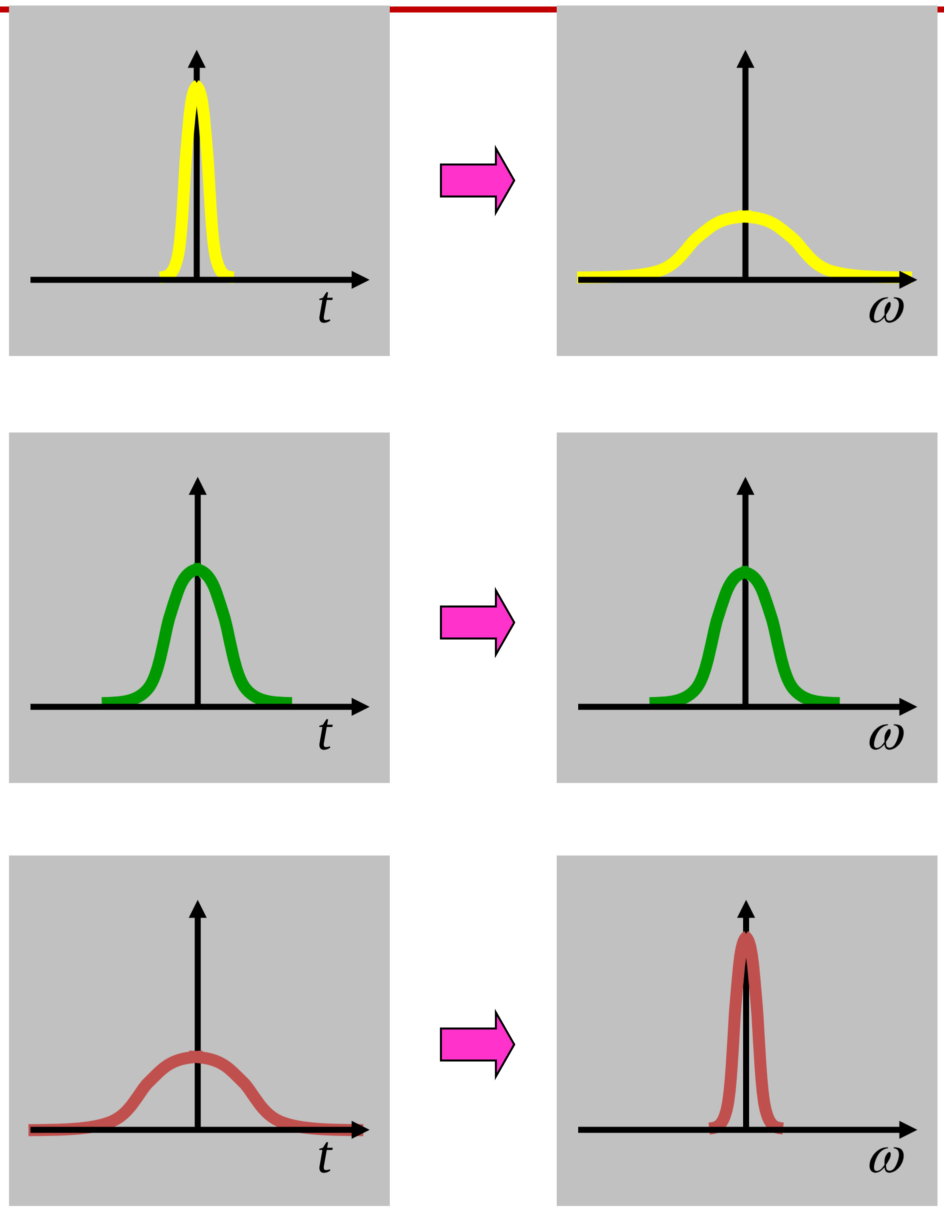
This is the essence of the Uncertainty Principle!

$f(t)$ $F(\omega)$

Short pulse

Medium-length pulse

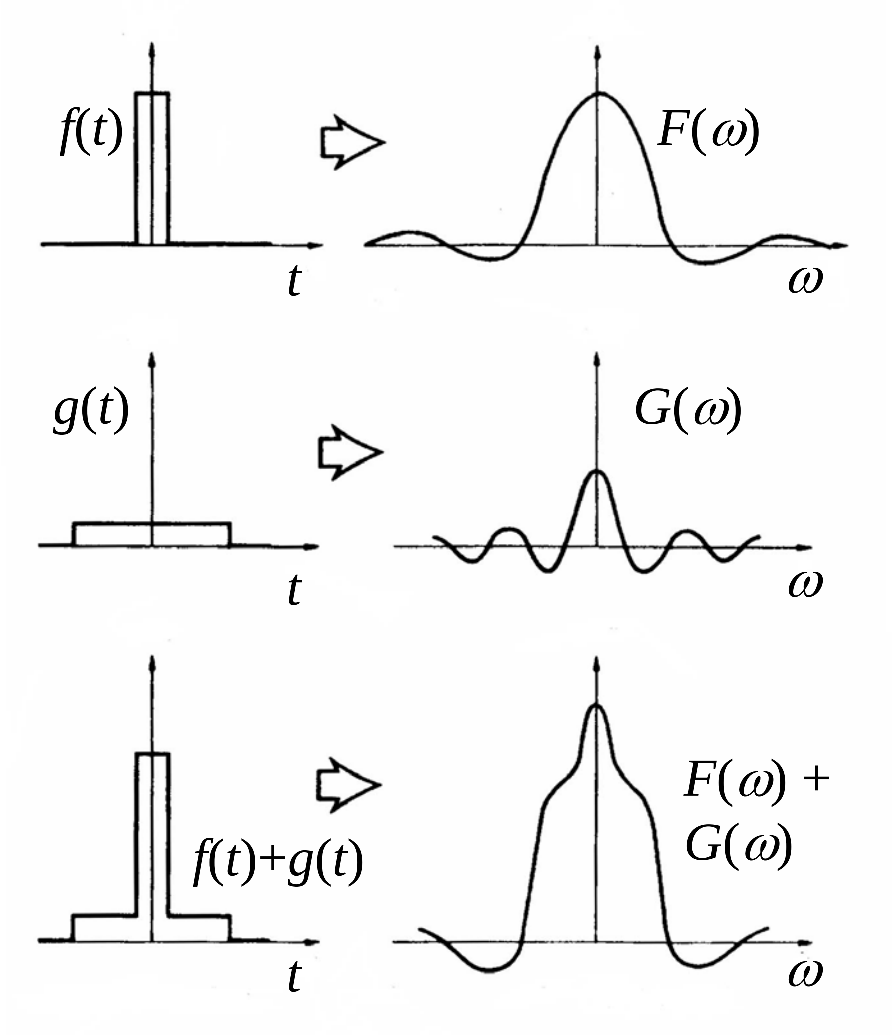
Long pulse



The Fourier Transform of a sum of two functions

$$\boxed{\begin{aligned}&\mathcal{F}\{a\,f(t) + b\,g(t)\} = \\ &a\mathcal{F}\{f(t)\} + b\mathcal{F}\{g(t)\}\end{aligned}}$$

Also, constants factor out.



Shift Theorem

The Fourier transform of a shifted function, $f(t-a)$:

$$\boxed{\mathcal{F}\left\{ f(t-a)\right\} = \exp(-i\omega a)F(\omega)}$$

Proof :

$$\mathcal{F}\left\{ f\left(t-a\right)\right\} = \int_{-\infty}^{\infty} f(t-a)\exp(-i\omega t)dt$$

Change variables : $u = t - a$

$$\int_{-\infty}^{\infty} f(u)\exp(-i\omega[u+a])du$$

$$= \exp(-i\omega a)\int_{-\infty}^{\infty} f(u)\exp(-i\omega u)du$$

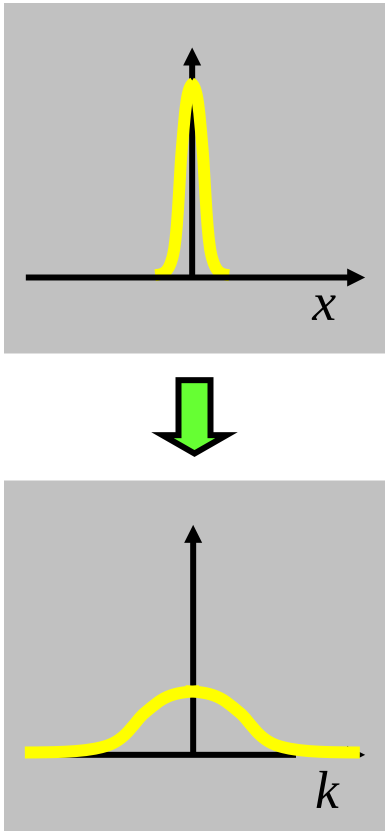
$$= \exp(-i\omega a)F(\omega)$$

Fourier Transform with respect to space

If $f(x)$ is a function of position,

$$\boxed{F(k) = \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx}$$

$$\mathscr{F}\{f(x)\} = F(k)$$



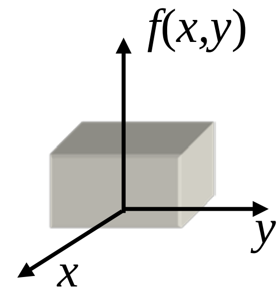
We refer to k as the **spatial frequency**.

Everything we've said about Fourier transforms between the t and ω domains also applies to the x and k domains.

The 2D Fourier Transform

$$\mathscr{F}^{(2)}\{f(x,y)\} \;=\; F(k_x,k_y)$$

$$= \iint f(x,y)\, \exp[-i(k_x x + k_y y)]\; dx\, dy$$

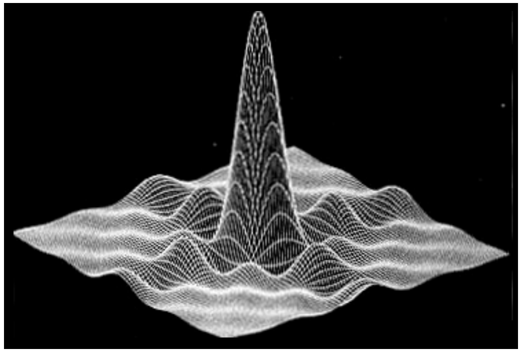


If $f(x,y) = f_x(x)\, f_y(y)$,

then the 2D FT splits into two 1D FT's.

But this doesn’t always happen.

$\mathscr{F}^{(2)}\{f(x,y)\}$



Fibers (will not be covered in 2010)

and

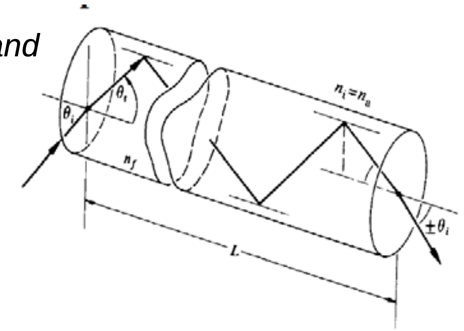


Figure 5.70 Rays reflected within a dielectric cylinder.

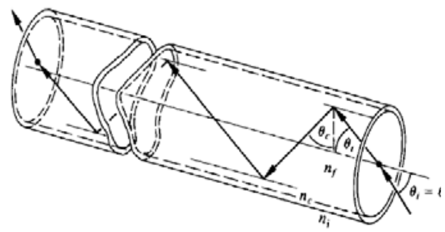


Figure 5.71 Rays in a clad optical fiber.

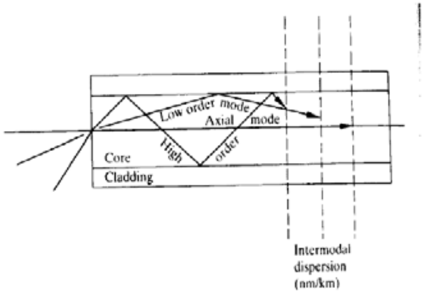


Figure 5.73 Intermodal dispersion in a stepped-index multimode fiber.

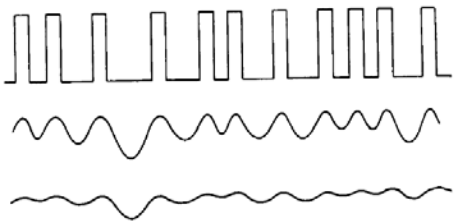


Figure 5.74 Rectangular pulses of light smeared out by increasing amounts of dispersion. Note how the closely spaced pulses degrade more quickly.

1. Total reflection.
2. Corning Glass Works, 1970: fiber with similar attenuation of copper cable. 1% per km, or 20 dB/km. Currently, 96% per km or better, i.e., 0.16 dB/km.
3. Benefit comparing to copper cables: low-loss, high data rate, small size and weight, immune to electromagnetic interference, low cost.
4.

Calculation of acceptance angle θ_{max} which is the maximum incident angle for a ray to experience total reflection in the fiber.

$$\theta_c = \frac{n_c}{n_f} = \sin(90^\circ - \theta_t)$$

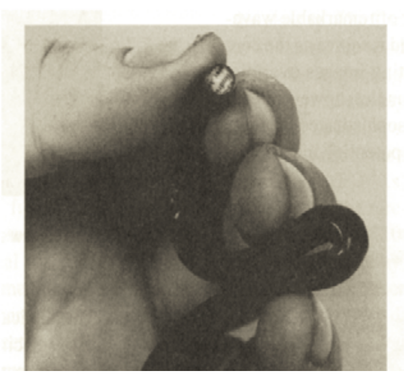
Thus,

$$\frac{n_c}{n_f} = \cos\theta_t = \sqrt{1-\sin^2\theta_t}$$

Applying Snell;s Law,

$$\sin\theta_{max} = \frac{1}{n_i}\sqrt{n_f^2 - n_c^2}$$

Numerical aperture (NA): $n_i \sin\theta_{max}$, the light-gathering power.



$$NA = \left(n_f^2 - n_c^2\right)^{1/2}$$

Example:

Let axial length be L, the shortest length of ray path. Then, the longest path L_{max} is when the incident angle is θ_c. The time difference Δt becomes

$$\Delta t = \frac{L_{max} - L}{v_f} = \frac{Ln_f^2}{cn_c} - \frac{Ln_f}{c} = \frac{Ln_f}{c}\left(\frac{n_f}{n_c} - 1\right)$$

If n_f=1.5 and n_c=1.489, then $\Delta t/L$=37 ns/km, or a separation of distance 7.4 m/km. In order to make the signal readable, the spatial separation might need to be twice of the spread-out width. If the line is 1 km long, the output pulse is 7.4 m long, the separation should be 14.8 m or 74 ns apart, which is 13.5 Million/s.

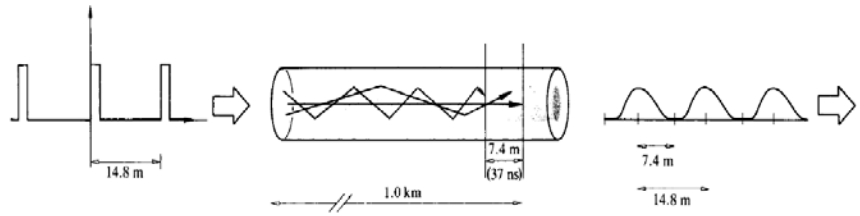


Figure 5.75 The spreading of an input signal due to intermodal dispersion.

The number of modes in a stepped-index fiber is

$$N_m \approx \tfrac{1}{2}\left(\pi D \times NA / \lambda_0\right)^2$$