

Gaussian Beam Optics

[Hecht Ch. 13.1 pages 594-596 Notes from Melles Griot and Newport]

Readings:

For details on the theory of Gaussian beam optics, refer to the excerpts from the Melles Griot and Newport catalogs. Melles Griot, along with Newport Corporation, is a major manufacturer of optical components used for research. You will also find tutorials on other subjects of optics and photonics at their web site:

<http://www.mellesgriot.com/products/optics/toc.htm> (CVI Melles Griot Optics Guides)

<http://www.newport.com/servicesupport/Tutorials/default.aspx?id=110> (Newport Tutorials).

Gaussian Beam Calculators:

There are many calculators or programs to determine Gaussian beam propagation in free space or through a combination of lenses. Below is a short list of program you may use to solve the homework problems.

Gaussian Beam Calculator for simple lenses

http://www.photonics.byu.edu/Gaussian_Beam_Propagation.phtml

<http://www.newport.com/OpticalAssistant/FocusCollimatedBeam.aspx>

<http://www.originalcode.com/downloads/GBC8p2.zip> (need LabView)

<http://www.novajo.ca/abcd/> (for Mac only, ABCD Gaussian Beam Propagation)

Gaussian Beam Intensity/Irradiance

http://www.calctool.org/CALC/phys/optics/gauss_power_dist

Gaussian Beam and **Spatial Filtering**

<http://www.newport.com/OpticalAssistant/SpatialFilterPinhole.aspx>

Introduction

One usually thinks of a laser beam as a perfectly collimated beam of light rays with the beam energy uniformly spread across the cross section of the beam. This is not an adequate picture for discussing the propagation of a laser beam over any appreciable distance because diffraction causes the light waves to spread transversely as they propagate, Fig. 1.

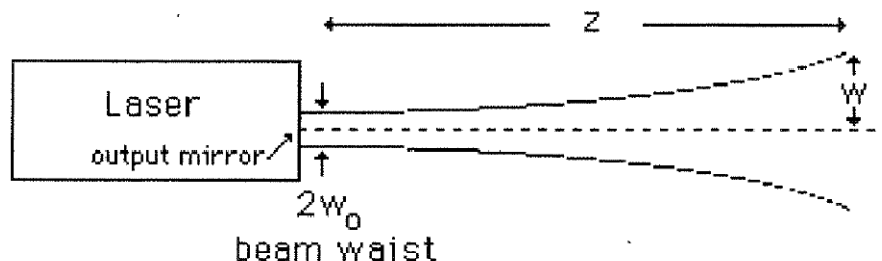


Figure 1. Divergence of a Laser Beam

Additionally, the energy (irradiance) profile of a laser beam is typically not uniform. For the most commonly used He-Ne lasers (operating in the TEM₀₀ mode) the irradiance (the power carried by the beam across a unit area perpendicular to the beam = W/m²) is given by a Gaussian function:

$$I(r) = I_0 e^{-2r^2/w^2} = \frac{2P}{\pi w^2} e^{-2r^2/w^2}, \quad (1)$$

where w is defined as the distance out from the center axis of the beam where the irradiance drops to $1/e^2$ of its value on axis. P is the total power in the beam. r is the transverse distance from the central axis. w depends on the distance z the beam has propagated from the beam waist. w_0 is the beam radius at the waist. [The beam waist is defined as the point where the beam wave front was last flat (as opposed to spherical at other locations).] For a hemispherical laser cavity such as the one used for the He-Ne laser used in the lab, the waist is located roughly at the output mirror. w_0 is related to w by

$$w(z) = w_0 \left[1 + \left(\frac{\lambda z}{\pi w_0^2} \right)^2 \right]^{1/2} = w_0 \left[1 + \left(\frac{z}{z_R} \right)^2 \right]^{1/2} \quad (2)$$

where we have defined a new parameter, called the Rayleigh range,

$$z_R = \frac{\pi w_0^2}{\lambda}, \quad (3)$$

which combines the wavelength and waist radius into a single parameter and completely describes the divergence of the Gaussian beam. Note that the Rayleigh range is the distance from the beam waist to the point at which the beam radius has increased to $\sqrt{2}w_0$. For a 633 nm red He-Ne laser with a waist of 0.4 mm, $z_R \approx 0.8$ m.

When $z \gg z_R$, Eq. (2) simplifies to $w = w_0 z / z_R$ and the laser beam diverges at a constant angle

$$\theta = \frac{w}{z} = \frac{w_0}{z_R} = \frac{\lambda}{\pi w_0} \quad (4)$$

Note that the smaller the Rayleigh range, the more rapidly the beam diverges.

Experimentally, one can use a CCD detector array to measure how the irradiance varies across the beam for several values of $z \gg z_R$. Then fit the data for each z using Eq. (1), which will yield values for $w(z)$. Then one can use Eqs. (3) and (4) to determine w_0 and calculate z_R .

To illustrate how to use Gaussian beam optics we will consider how to make a beam expander, which in its simplest form can simply be an astronomical telescope operated backward. The reason beam expanders are useful is that if you want to project a laser beam over a long distance -- such as for accurately measuring the distance to the moon where you time how long it takes a laser pulse to travel to the moon and back -- you do not want the beam to expand so much you can't detect it when it returns. We see from Eq. (4) that the smaller the waist radius, the more rapidly the beam diverges. Thus to project the beam you must first expand its waist radius to reduce the divergence at greater distances. To understand the beam expander we first review the operation of an astronomical telescope.

Astronomical telescope: In order to understand how a beam expander works, it is useful to review the astronomical telescope. The telescope, as shown in Fig. 2, uses a long focal length (f_o) objective lens to focus (parallel) rays from a distant object on its focal plane. The short focal length (f_e) eye piece lens is placed so that this image lies on its focal plane (i.e. the two lenses are separated by f_o+f_e). The rays reaching the eye are again parallel, but appear to subtend a much larger angle than the original object. From Fig. 2 it is easy to see that the angular magnification is

$$m = \frac{\theta_i}{\theta_o} = \frac{h/f_e}{h/f_o} = \frac{f_o}{f_e}.$$

Thus the astronomical telescope converts a narrowly diverging beam into a widely diverging one. By interchanging the object and eye we see that a diverging beam can be made to diverge less strongly.

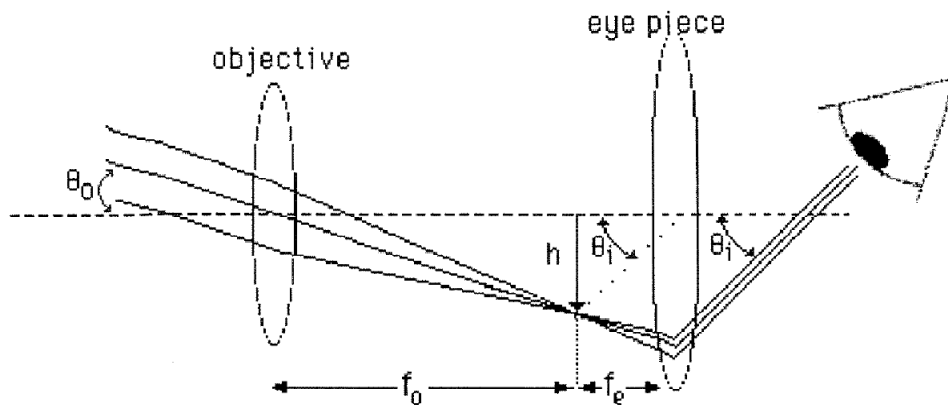


Figure 2. The Astronomical Telescope

Beam expander:

Because Gaussian beams do not follow the rules of ray optics, we cannot use the lens equation to design a beam expander. However, as discuss in the Melles Griot Optics Guide, if you consider the object to be the beam waist of the incoming beam and the image to be the beam waist after the beam passes through the lens, then you can use a modified lens equation:

$$\frac{1}{s/f + (z_R/f)^2 / (s/f - 1)} + \frac{1}{s''/f} = 1, \quad (5)$$

where s is the object (waist) distance from the lens, s'' is the image (waist) distance, and f is the focal length of the lens. You will need two additional equations for the magnification and the Rayleigh range of the image beam waist:

$$m = \frac{w_o''}{w_o} = \frac{1}{\sqrt{\left[1 - \frac{s}{f}\right]^2 + \left(\frac{z_R}{f}\right)^2}} \quad (6)$$

and

$$z_R'' = m^2 z_R. \quad (7)$$

If you have several lenses you apply Eqs. (5)-(7) to each lens in succession with the image of the first becoming the objection of the second, etc.

Lets' now apply this to an inverted astronomical telescope with the focal length of the first lens being 5 cm and the second 40 cm. In the astronomical telescope the two lenses are separated by the sum of the focal lengths of the two lenses – 45 cm in this case. We assume a red He-Ne laser (633nm) with beam waist radius of 0.4 mm. We first use Eq. (3) to get $z_R=0.80$ m. For the first lens, $f=5$ cm, and the beam waist for the laser is close to the exit of the laser. We put the lens as close to the laser as possible and assume $s=0$. The using Eq. (5) we get

$$\frac{s''}{f} = \frac{1}{1 + f^2/z_R^2} = 0.996.$$

[Notice the difference here with ordinary ray optics -- where would the image be if we put the object right at the lens?]

Thus it would seem we have simply moved the waist out by 5 cm. But note that the magnification is now

$$m = \frac{w_o''}{w_o} = \frac{1}{\sqrt{1 + \left(\frac{z_R}{f}\right)^2}} = 0.0613$$

or $w_o''=0.0245$ mm and $z_R''=0.300$ cm. Instead of expanding the beam we have compressed it, but now we will expand it with the second lens. We place the two lenses 45 cm apart so that the waist (s'') is at the focal point of the second lens. Then using $s = 40$ cm, or $s/f = 1$, in Eq. (5) we see the waist is again at the focal point of the lens (i.e. 40 cm in front of the second lens) and using Eq. (6) we find:

$$m'' = \frac{w_o''''}{w_o''} = \frac{f}{z_R} = 133$$

or $w_o'''' = 3.27$ mm and $z_R'''' = 5330$ cm. We then see from the definition of the Rayleigh range that after traveling through the expander, the beam will travel 53.3 m from the waist before the radius increases by a further $\sqrt{2}$ factor (to 4.62 mm). Or put another way, after traveling 1000 m, the expanded beam would have a beam radius of 61.4 mm while the original beam would have a radius of 500 mm. [We have used the asymptotic relation $z/z_R = w/w_o$.]

Gaussian Beam Propagation

In most laser applications it is necessary to focus, modify, or shape the laser beam by using lenses and other optical elements. In general, laser-beam propagation can be approximated by assuming that the laser beam has an ideal Gaussian intensity profile, which corresponds to the theoretical TEM₀₀ mode. Coherent Gaussian beams have peculiar transformation properties which require special consideration. In order to select the best optics for a particular laser application, it is important to understand the basic properties of Gaussian beams.

Unfortunately, the output from real-life lasers is not truly Gaussian (although helium neon lasers and argon-ion lasers are a very close approximation). To accommodate this variance, a quality factor, M^2 (called the “M-squared” factor), has been defined to describe the deviation of the laser beam from a theoretical Gaussian. For a theoretical Gaussian, $M^2 = 1$; for a real laser beam, $M^2 > 1$. The M^2 factor for helium neon lasers is typically less than 1.1; for ion lasers, the M^2 factor typically is between 1.1 and 1.3. Collimated TEM₀₀ diode laser beams usually have an M^2 ranging from 1.1 to 1.7. For high-energy multimode lasers, the M^2 factor can be as high as 25 or 30. In all cases, the M^2 factor affects the characteristics of a laser beam and cannot be neglected in optical designs.

In the following section, Gaussian Beam Propagation, we will treat the characteristics of a theoretical Gaussian beam ($M^2=1$); then, in the section Real Beam Propagation we will show how these characteristics change as the beam deviates from the theoretical. In all cases, a circularly symmetric wavefront is assumed, as would be the case for a helium neon laser or an argon-ion laser. Diode laser beams are asymmetric and often astigmatic, which causes their transformation to be more complex.

Although in some respects component design and tolerancing for lasers is more critical than for conventional optical components, the designs often tend to be simpler since many of the constraints associated with imaging systems are not present. For instance, laser beams are nearly always used on axis, which eliminates the need to correct asymmetric aberration. Chromatic aberrations are of no concern in single-wavelength lasers, although they are critical for some tunable and multiline laser applications. In fact, the only significant aberration in most single-wavelength applications is primary (third-order) spherical aberration.

Scatter from surface defects, inclusions, dust, or damaged coatings is of greater concern in laser-based systems than in incoherent systems. Speckle content arising from surface texture and beam coherence can limit system performance.

Because laser light is generated coherently, it is not subject to some of the limitations normally associated with incoherent sources. All parts of the wavefront act as if they originate from the same point; consequently, the emergent wavefront can be precisely defined. Starting out with a well-defined wavefront permits more precise focusing and control of the beam than otherwise would be possible.

For virtually all laser cavities, the propagation of an electromagnetic field, $E^{(0)}$, through one round trip in an optical resonator can be described mathematically by a propagation integral, which has the general form

$$E^{(1)}(x, y) = e^{-jkp} \iint_{\text{Input Plane}} K(x, y, x_0, y_0) E^{(0)}(x_0, y_0) dx_0 dy_0 \quad (2.1)$$

where K is the propagation constant at the carrier frequency of the optical signal, p is the length of one period or round trip, and the integral is over the transverse coordinates at the reference or input plane. The function K is commonly called the propagation kernel since the field $E^{(1)}(x, y)$, after one propagation step, can be obtained from the initial field $E^{(0)}(x_0, y_0)$ through the operation of the linear kernel or “propagator” $K(x, y, x_0, y_0)$.

By setting the condition that the field, after one period, will have exactly the same transverse form, both in phase and profile (amplitude variation across the field), we get the equation

$$\gamma_{nm} E_{nm}(x, y) \equiv \iint_{\text{Input Plane}} K(x, y, x_0, y_0) E_{nm}(x_0, y_0) dx_0 dy_0 \quad (2.2)$$

where E_{nm} represents a set of mathematical eigenmodes, and γ_{nm} a corresponding set of eigenvalues. The eigenmodes are referred to as transverse cavity modes, and, for stable resonators, are closely approximated by Hermite-Gaussian functions, denoted by TEM_{nm}. (Anthony Siegman, Lasers)

The lowest order, or “fundamental” transverse mode, TEM₀₀ has a Gaussian intensity profile, shown in figure 2.1, which has the form

$$I(x, y) \propto e^{-k(x^2 + y^2)} \quad (2.3)$$

In this section we will identify the propagation characteristics of this lowest-order solution to the propagation equation. In the next section, Real Beam Propagation, we will discuss the propagation characteristics of higher-order modes, as well as beams that have been distorted by diffraction or various anisotropic phenomena.

BEAM WAIST AND DIVERGENCE

In order to gain an appreciation of the principles and limitations of Gaussian beam optics, it is necessary to understand the nature of the laser output beam. In TEM₀₀ mode, the beam emitted from a laser begins as a perfect plane wave with a Gaussian transverse irradiance profile as shown in figure 2.1. The Gaussian shape is truncated at some diameter either by the internal dimensions of the laser or by some limiting aperture in the optical train. To specify and discuss the propagation characteristics of a laser beam, we must define its diameter in some way. There are two commonly accepted definitions. One definition is the diameter at which

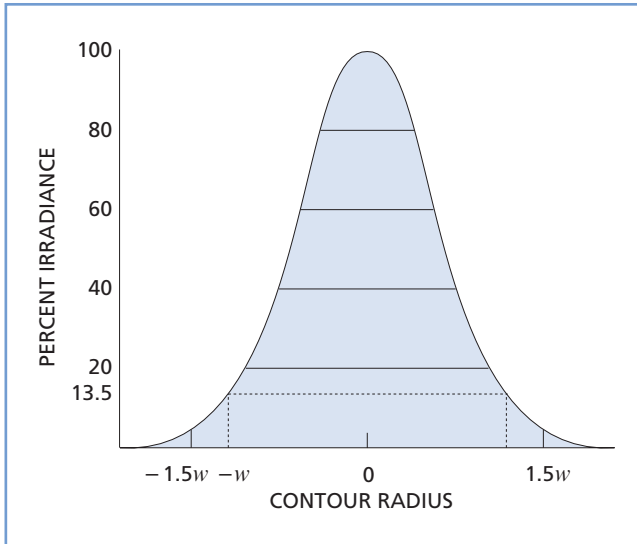


Figure 2.1 Irradiance profile of a Gaussian TEM₀₀ mode

the beam irradiance (intensity) has fallen to $1/e^2$ (13.5 percent) of its peak, or axial value and the other is the diameter at which the beam irradiance (intensity) has fallen to 50 percent of its peak, or axial value, as shown in figure 2.2. This second definition is also referred to as FWHM, or full width at half maximum. For the remainder of this guide, we will be using the $1/e^2$ definition.

Diffraction causes light waves to spread transversely as they propagate, and it is therefore impossible to have a perfectly collimated beam. The spreading of a laser beam is in precise accord with the predictions of pure diffraction theory; aberration is totally insignificant in the present context. Under quite ordinary circumstances, the beam spreading can be so small it can go unnoticed. The following formulas accurately describe beam spreading, making it easy to see the capabilities and limitations of laser beams.

Even if a Gaussian TEM₀₀ laser-beam wavefront were made perfectly flat at some plane, it would quickly acquire curvature and begin spreading in accordance with

$$R(z) = z \left[1 + \left(\frac{\pi w_0^2}{\lambda z} \right)^2 \right] \quad (2.4)$$

and

$$w(z) = w_0 \left[1 + \left(\frac{\lambda z}{\pi w_0^2} \right)^2 \right]^{1/2} \quad (2.5)$$

where z is the distance propagated from the plane where the wavefront is flat, λ is the wavelength of light, w_0 is the radius of the $1/e^2$ irradiance contour at the plane where the wavefront is flat, $w(z)$ is

the radius of the $1/e^2$ contour after the wave has propagated a distance z , and $R(z)$ is the wavefront radius of curvature after propagating a distance z . $R(z)$ is infinite at $z=0$, passes through a minimum at some finite z , and rises again toward infinity as z is further increased, asymptotically approaching the value of z itself. The plane $z=0$ marks the location of a Gaussian waist, or a place where the wavefront is flat, and w_0 is called the beam waist radius.

The irradiance distribution of the Gaussian TEM₀₀ beam, namely,

$$I(r) = I_0 e^{-2r^2/w^2} = \frac{2P}{\pi w^2} e^{-2r^2/w^2}, \quad (2.6)$$

where $w=w(z)$ and P is the total power in the beam, is the same at all cross sections of the beam.

The invariance of the form of the distribution is a special consequence of the presumed Gaussian distribution at $z=0$. If a uniform irradiance distribution had been presumed at $z=0$, the pattern at $z=\infty$ would have been the familiar Airy disc pattern given by a Bessel function, whereas the pattern at intermediate z values would have been enormously complicated.

Simultaneously, as $R(z)$ asymptotically approaches z for large z , $w(z)$ asymptotically approaches the value

$$w(z) = \frac{\lambda z}{\pi w_0} \quad (2.7)$$

where z is presumed to be much larger than $\pi w_0/\lambda$ so that the $1/e^2$ irradiance contours asymptotically approach a cone of angular radius

$$\theta = \frac{w(z)}{z} = \frac{\lambda}{\pi w_0}. \quad (2.8)$$

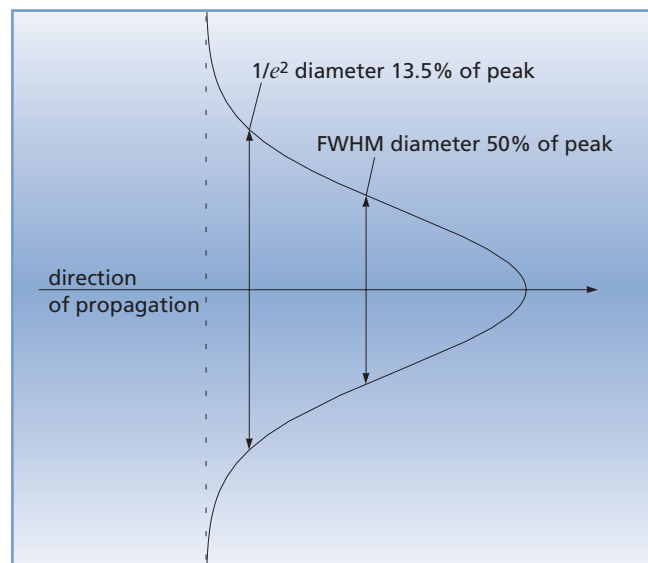


Figure 2.2 Diameter of a Gaussian beam

This value is the far-field angular radius (half-angle divergence) of the Gaussian TEM_{00} beam. The vertex of the cone lies at the center of the waist, as shown in figure 2.3.

It is important to note that, for a given value of λ , variations of beam diameter and divergence with distance z are functions of a single parameter, w_0 , the beam waist radius.

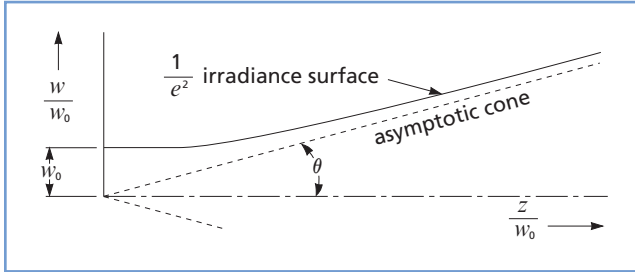


Figure 2.3 Growth in $1/e^2$ radius with distance propagated away from Gaussian waist

Near-Field vs Far-Field Divergence

Unlike conventional light beams, Gaussian beams do not diverge linearly. Near the beam waist, which is typically close to the output of the laser, the divergence angle is extremely small; far from the waist, the divergence angle approaches the asymptotic limit described above. The Rayleigh range (z_R), defined as the distance over which the beam radius spreads by a factor of $\sqrt{2}$, is given by

$$z_R = \frac{\pi w_0^2}{\lambda} \quad (2.9)$$

At the beam waist ($z=0$), the wavefront is planar [$R(0)=\infty$]. Likewise, at $z=\infty$, the wavefront is planar [$R(\infty)=\infty$]. As the beam propagates from the waist, the wavefront curvature, therefore, must increase to a maximum and then begin to decrease, as shown in figure 2.4. The Rayleigh range, considered to be the dividing line

between near-field divergence and mid-range divergence, is the distance from the waist at which the wavefront curvature is a maximum. Far-field divergence (the number quoted in laser specifications) must be measured at a distance much greater than z_R (usually $>10 \times z_R$ will suffice). This is a very important distinction because calculations for spot size and other parameters in an optical train will be inaccurate if near- or mid-field divergence values are used. For a tightly focused beam, the distance from the waist (the focal point) to the far field can be a few millimeters or less. For beams coming directly from the laser, the far-field distance can be measured in meters.

Typically, one has a fixed value for w_0 and uses the expression

$$w(z) = w_0 \left[1 + \left(\frac{\lambda z}{\pi w_0^2} \right)^2 \right]^{1/2}$$

to calculate $w(z)$ for an input value of z . However, one can also utilize this equation to see how final beam radius varies with starting beam radius at a fixed distance, z . Figure 2.5 shows the Gaussian beam propagation equation plotted as a function of w_0 , with the particular values of $\lambda = 632.8$ nm and $z = 100$ m.

The beam radius at 100 m reaches a minimum value for a starting beam radius of about 4.5 mm. Therefore, if we wanted to achieve the best combination of minimum beam diameter and minimum beam spread (or best collimation) over a distance of 100 m, our optimum starting beam radius would be 4.5 mm. Any other starting value would result in a larger beam at $z = 100$ m.

We can find the general expression for the optimum starting beam radius for a given distance, z . Doing so yields

$$w_0(\text{optimum}) = \left(\frac{\lambda z}{\pi} \right)^{1/2} \quad (2.10)$$

Using this optimum value of w_0 will provide the best combination of minimum starting beam diameter and minimum beam

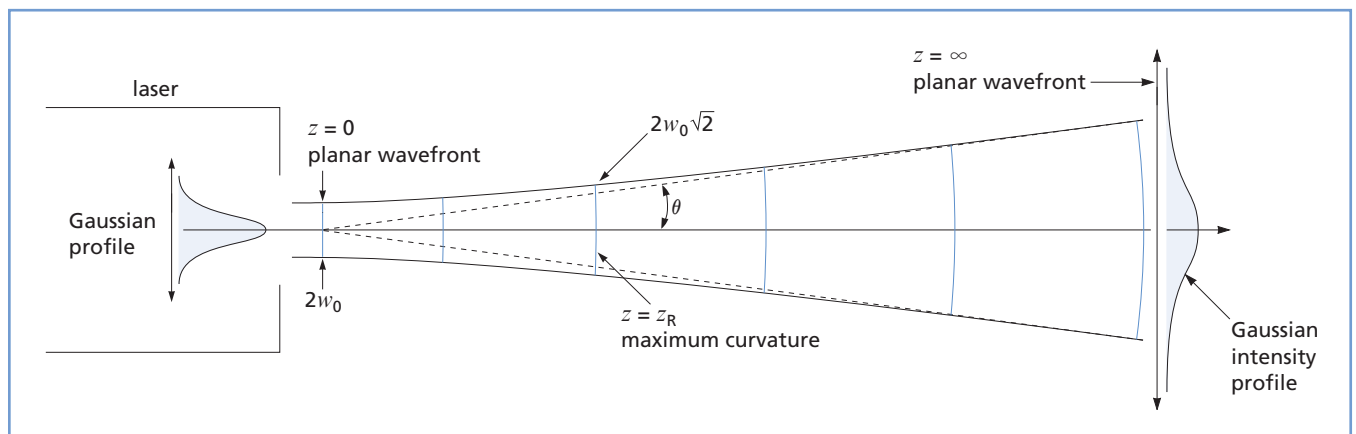


Figure 2.4 Changes in wavefront radius with propagation distance

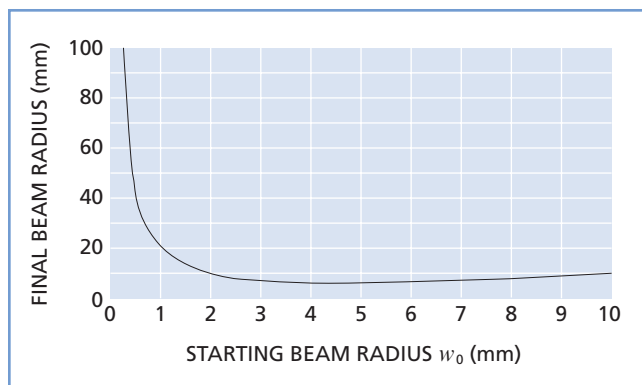


Figure 2.5 **Beam radius at 100 m as a function of starting beam radius for a HeNe laser at 632.8 nm**

spread [ratio of $w(z)$ to w_0] over the distance z . For $z = 100$ m and $\lambda = 632.8$ nm, w_0 (optimum) = 4.48 mm (see example above). If we put this value for w_0 (optimum) back into the expression for $w(z)$,

$$w(z) = \sqrt{2}(w_0) \quad (2.11)$$

Thus, for this example,

$$w(100) = \sqrt{2}(4.48) = 6.3 \text{ mm}$$

By turning this previous equation around, we find that we once again have the Rayleigh range (z_R), over which the beam radius spreads by a factor of $\sqrt{2}$ as

$$z_R = \frac{\pi w_0^2}{\lambda}$$

with

$$w(z_R) = \sqrt{2}w_0.$$

If we use beam-expanding optics that allow us to adjust the position of the beam waist, we can actually double the distance over which beam divergence is minimized, as illustrated in figure 2.6. By focusing the beam-expanding optics to place the beam waist at the midpoint, we can restrict beam spread to a factor of $\sqrt{2}$ over a distance of $2z_R$, as opposed to just z_R .

This result can now be used in the problem of finding the starting beam radius that yields the minimum beam diameter and beam spread over 100 m. Using $2(z_R) = 100$ m, or $z_R = 50$ m, and $\lambda = 632.8$ nm, we get a value of $w(z_R) = (2\lambda/\pi)^{1/2} = 4.5$ mm, and $w_0 = 3.2$ mm. Thus, the optimum starting beam radius is the same as previously calculated. However, by focusing the expander we achieve a final beam radius that is no larger than our starting beam radius, while still maintaining the $\sqrt{2}$ factor in overall variation.

Alternately, if we started off with a beam radius of 6.3 mm, we could focus the expander to provide a beam waist of $w_0 = 4.5$ mm at 100 m, and a final beam radius of 6.3 mm at 200 m.

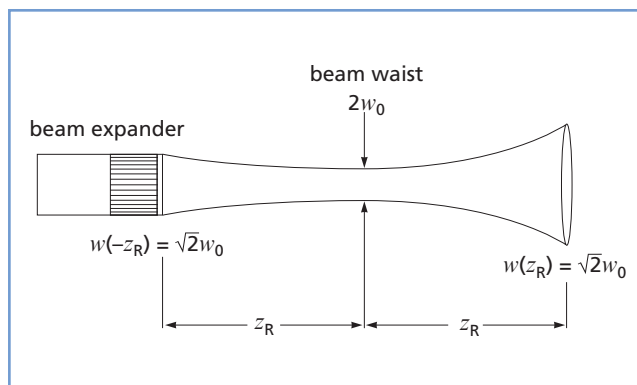


Figure 2.6 **Focusing a beam expander to minimize beam radius and spread over a specified distance**

APPLICATION NOTE

Location of the beam waist

The location of the beam waist is required for most Gaussian-beam calculations. Melles Griot lasers are typically designed to place the beam waist very close to the output surface of the laser. If a more accurate location than this is required, our applications engineers can furnish the precise location and tolerance for a particular laser model.

Do you need . . .

BEAM EXPANDERS

Melles Griot offers a range of precision beam expanders for better performance than can be achieved with the simple lens combinations shown here. Available in expansion ratios of $3 \times$, $10 \times$, $20 \times$, and $30 \times$, these beam expanders produce less than $\lambda/4$ of wavefront distortion. They are optimized for a 1-mm-diameter input beam, and mount using a standard 1-inch-32 TPI thread. For more information, see page 16.4.



Gaussian Beam Optics

The Gaussian is a radially symmetric distribution whose electric field variation is given by:

$$E_s = E_0 \exp\left(-\frac{r^2}{\omega_0^2}\right)$$

Its Fourier transform is also a Gaussian distribution. If we were to solve the Fresnel integral itself rather than the Fraunhofer approximation, we would find that a Gaussian source distribution remains Gaussian at every point along its path of propagation through the optical system. This makes it particularly easy to visualize the distribution of the fields at any point in the optical system. The intensity is also Gaussian:

$$I_s = \eta E_s E_s^* = \eta E_0 E_0^* \exp\left(-\frac{2r^2}{\omega_0^2}\right)$$

This relationship is much more than a mathematical curiosity, since it is now easy to find a light source with a Gaussian intensity distribution: the laser. Most lasers automatically oscillate with a Gaussian distribution of electric field. The basic Gaussian may also take on some particular polynomial multipliers and still remain its own transform. These field distributions are known as higher-order transverse modes and are usually avoided by design in most practical lasers.

The Gaussian has no obvious boundaries to give it a characteristic dimension like the diameter of the circular aperture, so the definition of the size of a Gaussian is somewhat arbitrary. Figure 1 shows the Gaussian intensity distribution of a typical HeNe laser.

$$I(r) = I_0 \exp\left(-\frac{2r^2}{\omega_0^2}\right)$$

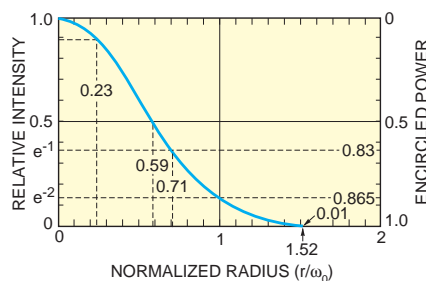


Figure 1

The parameter ω_0 , usually called the Gaussian beam radius, is the radius at which the intensity has decreased to $1/e^2$ or 0.135 of its axial, or peak value. Another point to note is the radius of half maximum, or 50% intensity, which is $0.59\omega_0$. At $2\omega_0$, or twice the Gaussian radius, the intensity is 0.0003 of its peak value, usually completely negligible.

The power contained within a radius r , $P(r)$, is easily obtained by integrating the intensity distribution from 0 to r :

$$P(r) = P(\infty) \left[1 - \exp\left(-\frac{2r^2}{\omega_0^2}\right) \right]$$

When normalized to the total power of the beam, $P(\infty)$ in watts, the curve is the same as that for intensity, but with the ordinate inverted. Nearly 100% of the power is contained in a radius $r = 2\omega_0$. One-half the power is contained within $0.59\omega_0$, and only about 10% of the power is contained with $0.23\omega_0$, the radius at which the intensity has decreased by 10%. The total power, $P(\infty)$ in watts, is related to the on-axis intensity, $I(0)$ (watts/m²), by:

$$P(\infty) = \left(\frac{\pi\omega_0^2}{2}\right) I(0)$$

$$I(0) = P(\infty) \left(\frac{2}{\pi\omega_0^2}\right)$$

The on-axis intensity can be very high due to the small area of the beam.

Care should be taken in cutting off the beam with a very small aperture. The source distribution would no longer be Gaussian, and the far-field intensity distribution would develop zeros and other non-Gaussian features. However, if the aperture is at least three or four ω_0 in diameter, these effects are negligible.

Propagation of Gaussian beams through an optical system can be treated almost as simply as geometric optics. Because of the unique self-Fourier Transform characteristic of the Gaussian, we do not need an integral to describe the evolution of the intensity profile with distance. The transverse distribution intensity remains Gaussian at every point in the system; only the radius of the Gaussian and the radius of curvature of the wavefront change. Imagine that we somehow create a coherent light beam with a Gaussian distribution and a plane wavefront at a position $x=0$. The beam size and wavefront curvature will then vary with x as shown in Figure 2.

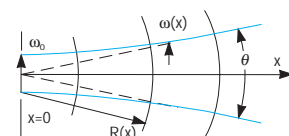


Figure 2

The beam size will increase, slowly at first, then faster, eventually increasing proportionally to x . The wavefront radius of curvature, which was infinite at $x = 0$, will become finite and initially decrease with x . At some point it will reach a minimum value, then increase with larger x , eventually becoming proportional to x . The equations describing the Gaussian beam radius $w(x)$ and wavefront radius of curvature $R(x)$ are:

$$\omega^2(x) = \omega_0^2 \left[1 + \left(\frac{\lambda x}{\pi \omega_0^2} \right)^2 \right]$$

$$R(x) = x \left[1 + \left(\frac{\pi \omega_0^2}{\lambda x} \right)^2 \right]$$

where ω_0 is the beam radius at $x = 0$ and λ is the wavelength. The entire beam behavior is specified by these two parameters, and because they occur in the same combination in both equations, they are often merged into a single parameter, x_R , the Rayleigh range:

$$x_R = \frac{\pi \omega_0^2}{\lambda}$$

In fact, it is at $x = x_R$ that R has its minimum value.

Note that these equations are also valid for negative values of x . We only imagined that the source of the beam was at $x = 0$; we could have created the same beam by creating a larger Gaussian beam with a negative wavefront curvature at some $x < 0$. This we can easily do with a lens, as shown in Figure 3.

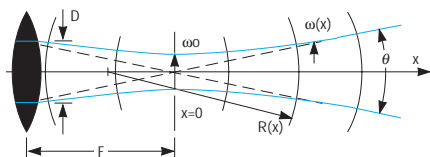


Figure 3

The input to the lens is a Gaussian with diameter D and a wavefront radius of curvature which, when modified by the lens, will be $R(x)$ given by the equation above with the lens located at $-x$ from the beam waist at $x = 0$. That input Gaussian will also have a beam waist position and size (or Rayleigh range) associated with it. Thus we can generalize the law of propagation of a Gaussian through even a complicated optical system.

In the free-space between lenses, mirrors and other optical elements, the position of the beam waist and the waist diameter (or Rayleigh range) completely describe the beam. When a beam passes through a lens, mirror, or dielectric interface, the diameter is unchanged but the wavefront curvature is changed, resulting in new values of waist position and waist diameter (or Rayleigh range) on the output side of the interface.

These equations, with input values for ω and R , allow the tracing of a Gaussian beam through any optical system with some restrictions: optical surfaces need to be spherical and with not-too-short focal lengths, so that beams do not change diameter too fast. These are exactly the analog of the paraxial restrictions used to simplify geometric optical propagation.

It turns out that we can put these laws in a form as convenient as the ABCD matrices used for geometric ray tracing. But there is a difference: $\omega(x)$ and $R(x)$ do not transform in matrix fashion as r and u do for ray tracing; rather, they transform via a complex bi-linear transformation:

$$q_{\text{out}} = \frac{q_{\text{in}} A + B}{q_{\text{in}} C + D}$$

where the quantity q is a complex composite of ω and R :

$$\frac{1}{q(x)} = \frac{1}{R(x)} - \frac{j\lambda}{\pi w(x)^2}$$

We can see from the expression for q that at a beam waist ($R = \infty$ and $\omega = \omega_0$), q is pure imaginary and equals jx_R . If we know where one beam waist is and its size, we can calculate q there and then use the bilinear ABCD relation to find q anywhere else. To determine the size and wavefront curvature of the beam everywhere in the system, you would use the ABCD values for each element of the system and trace q through them via successive bilinear transformations. But if you only wanted the overall transformation of q , you could multiply the elemental ABCD values in matrix form, just as is done in geometric optics, to find the overall ABCD values for the system, then apply the bilinear transform. For more information about Gaussian beams, see Chapter 17 of Siegman's book, *Lasers*.

Fortunately, simple approximations for spot size and depth of focus can still be used in most optical systems to select pinhole diameters, couple light into fibers, or compute laser intensities. Only when f -numbers are large should the full Gaussian equations be needed.

At large distances from a beam waist, the beam appears to diverge as a spherical wave from a point source located at the center of the waist. Note that "large" distances mean where $x \gg x_R$ and are typically very manageable considering the small area of most laser beams. The diverging beam has a full angular width θ (again, defined by $1/e^2$ points):

$$\theta = \frac{4\lambda}{2\pi w_0}$$

We have invoked the approximation $\tan\theta \approx \theta$ since the angles are small. Since the origin can be approximated by a point source, θ is given by geometrical optics as the diameter illuminated on the lens, D , divided by the focal length of the lens.

$$\theta \approx \frac{D}{F} = (f/\#)^{-1}$$

where $f/\#$ is the photographic f -number of the lens.

Equating these two expressions allows us to find the beam waist diameter in terms of the input beam parameters (with some restrictions that will be discussed later):

$$2w_0 = \left(\frac{4\lambda}{\pi}\right)\left(\frac{F}{D}\right)$$

We can also find the depth of focus from the formulas above. If we define the depth of focus (somewhat arbitrarily) as the distance between the values of x where the beam is $\sqrt{2}$ times larger than it is at the beam waist, then using the equation for $\omega(x)$ we can determine the depth of focus:

$$\text{DOF} = \left(\frac{8\lambda}{\pi}\right)\left(\frac{F}{D}\right)^2$$

Using these relations, we can make simple calculations for optical systems employing Gaussian beams. For example, suppose that we use a 10 mm focal length lens to focus the collimated output of a helium-neon laser (632.8 nm) that has a 1 mm diameter beam. The diameter of the focal spot will be:

$$\left(\frac{4}{\pi}\right)(632.8 \text{ nm})\left(\frac{10 \text{ mm}}{1 \text{ mm}}\right),$$

or about 8 μm . The depth of focus for the beam is then:

$$\left(\frac{8}{\pi}\right)(632.8 \text{ nm})\left(\frac{10 \text{ mm}}{1 \text{ mm}}\right)^2,$$

or about 160 μm . If we were to change the focal length of the lens in this example to 100 mm, the focal spot size would increase 10 times to 80 μm , or 8% of the original beam diameter. The depth of focus would increase 100 times to 16 mm.

However, suppose we increase the focal length of the lens to 2,000 mm. The “focal spot size” given by our simple equation would be 200 times larger, or 1.6 mm, 60% larger than the original beam! Obviously, something is wrong. The trouble is not with the equations giving $\omega(x)$ and $R(x)$, but with the assumption that the beam waist occurs at the focal distance from the lens. For weakly focused systems, the beam waist does not occur at the focal length. In fact, the position of the beam waist changes contrary to what we would expect in geometric optics: the waist moves toward the lens as the focal length of the lens is increased. However, we could easily believe the limiting case of this behavior by noting that a lens of infinite focal length such as a flat piece of glass placed at the beam waist of a collimated beam will produce a new beam waist not at infinity, but at the position of the glass itself.