How the universe expands —9 Feb 2010

Friedman's equation relates the expansion parameter $a(t)$, matter density $\rho(t)$, and the radius of curvature r_0. The position of a galaxy at comoving coordinate r is $r\,a(t)$.

$$\left(\frac{1}{a}\,\frac{d\,a}{d\,H_0\,t}\right)^2 - \frac{8\,\pi\,G}{3\,H_0^2}\,\rho = -\frac{1}{H_0^2\,r_0^2\,a^2}$$

Outline

• Interpret Friedman's equation and dE = −pdV as the force of gravity. What replaces Newton's $F = -G\,M\,m/X^2$?

• Measuring the universe

The expansion parameter is observed directly. (The univserse expands by the same factor as the wavelength of light.)

Measuring involves distances and angles, which involve $(t,\ r,\ \theta,\ \phi)$.

We need to derive $r(a)$, the comoving coordinate of a galaxy seen at expansion parameter a.

What replaces Newton's law of gravity when applied to the expansion of the universe?

What replaces Newton's law of gravity when applied to the expansion of the universe? We already know what it is for the motion of planets areound a star: Add a term $-M/r \times l^2/r^2$ to the energy.

Recall Friedman's equation

$$\left(\frac{d\,a}{d\,t}\right)^2 - \frac{8\pi G}{3}\rho\,a^3\,a^{-1} = -\frac{1}{r_0^2},$$

which we derived partly by considering the energy of a galaxy at some distance from us.

Recall the First Law of thermodynamics

$$d\,E = -p\,d\,V,$$

which says the energy in some region changed by the work done on it.

Rewrite

$$d\,\rho\,a^3 = -p\,d\,a^3$$

Differentiate F's equation

$$2\,\frac{d\,a}{d\,t}\,\frac{d\,a}{d\,t^2} = \frac{8\pi G}{3}\left(a^{-1}\,\frac{d}{d\,t}\,\rho\,a^3 - a^{-2}\,\frac{d\,a}{d\,t}\,\rho\,a^3\right)$$

Insert 1st law to get

$$\frac{d^2\,a}{d\,t^2}\,\frac{d\,a}{d\,t} = \frac{4\pi G}{3}\left(-p\,a^{-1}\,\frac{d}{d\,t}\,a^3 - a^{-2}\,\frac{d\,a}{d\,t}\,\rho\,a^3\right)$$

$$\frac{d^2\,a}{d\,t^2} = -G\,\frac{4\pi a^3}{3}\,(\rho + 3\,p)\,\frac{1}{a^2}$$

Since a galaxy is at fixed r, its distance from us is $x = r\,a$.

$$\frac{d^2\,x}{d\,t^2} = -G\,\frac{4\pi x^3}{3}\,(\rho + 3\,p)\,\frac{1}{x^2}$$

In words, the acceleration of gravity is $-G\,M/x^2$. What is M?

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```
Needs["PhysicalConstants`"]

Convert[StefanConstant / SpeedOfLight^3, Kilogram / Meter^3 / Kelvin^4]
```

Convert::temp :
Warning: Convert[old,new] converts units of temperature. ConvertTemperature[temp,old,new] converts absolute temperature. ≫

$$\frac{2.10451\times10^{-33}\ \text{Kilogram}}{\text{Kelvin}^4\ \text{Meter}^3}$$

What replaces Newton's law of gravity for pressureless matter?

$$\frac{d^2 x}{d t^2} = -G \frac{4 \pi x^3}{3} (\rho + 3 p) \frac{1}{x^2}$$

Pressureless matter means the speeds are much less than the speed of light.

$\frac{4 \pi x^3}{3} \rho$ is the mass in a sphere. (Newton found that the force exerted by a spherically distributed extended mass is the same as if all of the mass were placed at the center of the sphere.) The force of gravity on mass m is

$$F = -G (V \rho) m / x^2$$

What if the matter has some pressure? We can guess. The "mass" density is should be replaced by the mass-energy density. A particle of mass μ moving with speed v has energy

$$\mu \left(1 - v^2\right)^{-1/2}$$

The find the mass-energy density, add up $\mu \left(1 - v^2\right)^{-1/2}$ for all of the particles.

$$\rho = n \mu \left(1 - v^2\right)^{-1/2}.$$

Now we have to find the pressure. The pressure in the x-direction is the transport of x-momentum in the x-direction. The momentum transfer in $d t$ over area A is

$$n \frac{\mu v_x}{\left(1-v^2\right)^{1/2}} (A v_x d t)$$

Therefore the pressure is

$$p = n \frac{\mu v_x^2}{\left(1-v^2\right)^{1/2}}$$

If the system is isotropic,

$$p = \frac{1}{3} n \frac{\mu v^2}{\left(1-v^2\right)^{1/2}}$$

Then the force is

$$F = -G V n \mu\left(1 + v^2\right) \left(1 - v^2\right)^{-1/2} m / x^2$$

Q: If the galaxies in the universe are hot, the gravity is bigger. Why?

What replaces Newton's law of gravity for radiation?

$$\frac{d^2 x}{d^2 t} = -G \frac{4 \pi x^3}{3} (\rho + 3 p) \frac{1}{x^2}$$

For thermal radiation, the energy density is $4 \sigma / c\, T^4$, and the mass density is

$$\rho = 4 \sigma \big/ c^3\, T^4$$
$$= \; 2.1 \times 10^{-33}\ \text{kg m}^{-3}\ \text{K}^{-4}\ T^4$$

The pressure is

$$p = \frac{1}{3} \rho.$$

The energy density is $u = \rho c^2$, and n conventional units, $p_{\text{cu}} = p c^2$.

$$\frac{d^2 x}{d \;\; t^2} = -G \frac{4 \pi x^3}{3} (\rho + 3 p) \frac{1}{x^2}$$
$$F = -G V (2 \rho) m \frac{1}{x^2}$$

What is unusual about this force of gravity?

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What replaces Newton's law of gravity for the vacuum?

$$\frac{d^2 x}{d t^2} = -G \frac{4 \pi x^3}{3} (\rho + 3 p) \frac{1}{x^2}$$

The vacuum has energy density from the quantum mechanical principle $\Delta E \Delta t > h$. Each field has some "vacuum energy." This energy density is a property of the field and therefore does not depend on the expansion of the universe.

Recall $p = -\rho$.

$$F = -G V (\rho + 3 p) m \frac{1}{x^2}$$
$$= \ +G V (2 \rho) m \frac{1}{x^2}$$

What is unusual about this force?

Radial coordinate of a galaxy

A galaxy emits some light when the expansion parameter was a, and we see the light. What is its radial coordinate?
The metric is

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + a(t)^2 \left[d\, r^2 \big/ \left(1 - (r/r_0)^2\right) + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\right].$$

For the light pulse, ds = 0.

$$\int_{\text{emit}}^{\text{recep}} a(t)^{-1}\, d\, t = r_0 \int_0^{r/r_0} \left(1 - x^2\right)^{-1/2} d\, x.$$

RHS is

$$r_0 \arcsin(r/r_0)$$

where $(r_0\, H_0)^{-2} = \Omega_0 - 1$.
Work on LHS:
Recall the density parameter $\Omega_0 = \frac{8\pi G}{3 H_0^2}$. Scale time by the Hubble time $\tau = H_0\, t$

$$\left(\frac{d\,a}{d\,H_0\,t}\right)^2 - \frac{8\pi G}{3 H_0^2}\, \rho\, a^2 = -\frac{1}{H_0^2\, r_0^2}$$

$$\left(\frac{d\,a}{H_0\,d\,t}\right)^2 - \Omega_0\, \frac{\rho}{\rho_0}\, a^2 = 1 - \Omega_0$$

For pressureless matter, $\rho = \rho_0\, a^{-3}$.

$$\left(\frac{d\,a}{H_0\,d\,\tau}\right)^2 - \Omega_0\, a^{-1} = 1 - \Omega_0$$

$$a(t)^{-1}\, d\, t = a(t)^{-1}\, \frac{d\,t}{d\,a}\, d\, a$$

In general this must be integrated numerically. Let's find it for pressureless matter.
Put in Friedman's eqn. to get

$$\text{LHS} = H_0^{-1}\left[a^2 (1 - \Omega_0) + \Omega_0\, a\right]^{-1/2} d\, a$$

LHS is going to be arcsin of something. Let us do three cases.

$\Omega_0 = 1$

$$\text{LHS} = H_0^{-1} \int_a^1 x^{-1/2}\, d\, x$$

$$= \; 2\, H_0^{-1}\left(1 - a^{1/2}\right)$$

$$\text{RHS} = r_0 \arcsin(r/r_0) \to r \text{ since } r_0 \to \infty$$

Result

$$r(a) = 2\, H_0^{-1}\left(1 - a^{1/2}\right)$$

$\Omega_0 = 0$

$$r_0 = i\, H_0^{-1}$$

$$\text{RHS} = i\, H_0^{-1} \arcsin\left[r \big/ \left(i\, H_0^{-1}\right)\right]$$

$$\text{LHS} = H_0^{-1} \int_a^1 a^{-1}\, d\, a$$

$$= \; -H_0^{-1} \log a$$

Collect LHS=RHS to get

$$\iota \log a = \arcsin\left[r \big/ \left(i\, H_0^{-1}\right)\right]$$

$$\frac{1}{2\, i}\left(e^{i^2 \log a} - e^{-i^2 \log a}\right) = \frac{r}{i\, H_0^{-1}}$$

$$r(a) = \frac{1}{2}\, H_0^{-1}\left(a^{-1} - a\right)$$

$\mathbf{\Omega_0 = 2}$ $r_0 = H_0^{-1}$

$$r(a) = H_0^{-1}(1 - a)$$

In[97]:=
```
r1[a_] := 2 (1 - a^(1/2));
r0[a_] := 1/2 (a^-1 - a);
r2[a_] := 1 - a;
```

In[111]:=
```
fig[] := LogLogPlot[{.01/(r1[(1 + z)^-1] (1 + z)^-1), .01/(r0[(1 + z)^-1] (1 + z)^-1), .01/(r2[(1 + z)^-1] (1 + z)^-1), .01 / z},
    {z, .01, 10}, AxesLabel → {"z", "θ"},

    Epilog → {Text["Ω0=1", Log[#]] &@{2, .01 × 3 / r1[1 / 3]}, Text["Ω0=0", Log[#]] &@
       {2, .01 × 3 / r0[1 / 3]}, Text["Ω0=2", Log[#]] &@{2, .01 × 3 / r2[1 / 3]}},
    PlotRange → {.02, 1}, BaseStyle → {FontFamily → "Helvetica", FontSize → Medium}];
```

Angle subtended by a ruler

We observe a ruler at a great distance. What is the angle between the ends of the ruler? (This observation has been made by WMAP: The ruler was c×(the age of the universe at recombination).)

Consider two events:
Event A: One end of the ruler emits some light at time t_1. The comoving coordinate is r_1.
Event B: Other end of the ruler emits some light at time t_1.

The path for the two light pulses: θ is constant and r changes from r_1 to 0.

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + a(t)^2 \left[d\, r^2 \middle/ \left(1 - (r/r_0)^2\right) + r^2\left(d\,\theta^2 + \sin^2\theta\, d\,\phi^2\right)\right]$$

The distance between the two events is L, the length of the ruler.

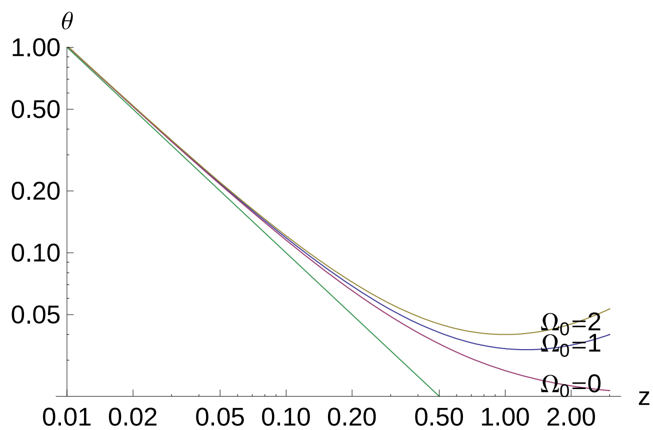
$$L = a(t_1)\, r_1\, \theta$$

Drop the subscripts, since it is clear a and r refer to the epoch and location of the ruler emitting the pulse of light.

$$\theta = \frac{L}{a\, r(a)}$$

In[108]:= `fig[]`

Out[108]=



Caption: Angle subtended by a ruler of length 0.01 H_0^{-1} The green line is $\theta = L\big/\left(z\, H_0^{-1}\right)$.

Q: Why do all lines match $\theta = L/(z\, H_0)$ for $z \ll 1$?
Q: Why is there a minimum angle for some cases?

In[34]:=
```
ta[η_, Ω_] := 1/2 Ω (Ω - 1)^(-3/2) {(η - Sin[η]), (1 - Cos[η])}
```

In[53]:=
```
fig[] := ParametricPlot[ta[η, 2], {η, 0, 2 π},
   Epilog → {Text["Now", ta[π / 2, 2], {-1.5, 0}], Point[ta[π / 2, 2]]},
   BaseStyle → {FontFamily → "Helvetica", FontSize → Medium}, AxesLabel → {"H₀t", "a"}]
```