Sound waves before recombination
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Physical conditions at recombination

At recombination, which has the greater mass density, pressureless matter or radiation?

$\Omega_{m0} = 0.26$ pressureless matter, mostly dark matter, matter that does not interact with light
$\Omega_{b0} = 0.043$ baryons, ordinary matter
$\Omega_{r0} = 1.2\times 10^{-5}$ radiation

$\rho_b = \rho_{b0}\, a^{-3}$
$\rho_r = \rho_{r0}\, a^{-4}$

$\rho_m / \rho_r = \rho_{m0} / \rho_{r0}\, a$
At $a = 0.000044$, the mass-energy density of pressureless matter and radiation are equal.
Q: At recombination, which has greater mass density, pressureless matter or radiation?

We will discuss sound waves, which has to do with radiation and matter.
Q: Does dark matter participate in the sound waves?

At $a_{eq} = 0.00028$ ($z = 3600$), the mass-energy density of baryonic matter and radiation are equal.
Q: At recombination, are electrons pressureless? The energy of a CBR photon is 2.3×10^{-4} eV.

At recombination, which has greater number density, baryonic matter or radiation?

At the present time, the mass of baryonic matter is 938MeV.
The mass of a photon is

$$2.73\, K / (11\,600\, K/\text{eV}) = 2.3\times 10^{-4}\ \text{eV}.$$

The number density

$$n_r / n_b = 0.00028 \times 938\ \text{MeV} / 2.3\times 10^{-4}\ \text{eV} = 1.1\times 10^9.$$

More precisely, becuase photons have different energies, I need to integrate the Planck number spectrum.

$$n_r = 0.41\times 10^9\ \text{photon}\ m^{-3} (T / 2.725\ \text{K})^3$$

$$n_b = 0.25\ \text{nucleon m}^{-3} (\Omega_{b0} / .043)\, (H_0 / 72\ \text{km/s/Mpc})^2$$

$$n_r / n_b = 1.64\times 10^9.$$

The number of photons and baryons do not change. As the universe expands, the number of baryons in a coexpanding box does not change. The number of baryons entering must equal the number exiting, because of homogeneity. Same argument is true for photons.
Q: The number of photons and baryons in this room do change when I turn on the light. In what sense is the previous statement correct?

Sound speed before recombination

The temperature at recombination is 3000K
There are many photons for every baryon or electron.

$n_r / n_b = 1.64\times 10^9$

At $a_{eq} = 0.00028$ ($z = 3600$), the mass-energy density of baryonic matter and radiation are equal.

The speed of sound

$$v_s = \left(\frac{d P}{d \rho}\right)^{1/2}$$

where the derivative is for adiabatic changes.

Proof: Newton's 2nd law $F = m\,a$ determines the movement of a sound wave, The force is due to an excess pressure. The mass is due to an excess density. Consider a slab of gas between x and $x + d\,x$. Because of the presence of the disturbance, x moves to $x + \chi(t, x)$. The $m\,a$ term becomes

$$(\rho_0\, d\,x)\,\frac{\partial^2 \chi}{\partial t^2}.$$

The force comes from the difference in pressure. The force term is

$$-\frac{\partial P}{\partial x}\, d\,x.$$

I need to relate pressure to mass density:

$$P = -\frac{\partial P}{\partial \rho}\,\rho_0\,\frac{\partial \chi}{\partial x}$$

Collect all; cancel $\rho_0$and $d\,x$:

$$\frac{\partial^2 \chi}{\partial t^2} = \frac{\partial P}{\partial \rho}\,\frac{\partial^2 \chi}{\partial x^2}$$

The speed of sound is

$$v_s = \left(\frac{d P}{d \rho}\right)^{1/2}$$

The derivative is taken with no heat flow, if the wavelength is large compared to the mean-free path.

Q: Just before recombination, does dark matter participate in the sound waves?

- **Values**

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ρCritical = .72^2 1.88*^-29 Gram / (Centi Meter)^3
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$$\frac{9.74592\times 10^{-30}\ \text{Gram}}{\text{Centi}^3\ \text{Meter}^3}$$

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Convert[(2.1045128753193778`*^-33 Kilogram) / (Kelvin^4 Meter^3) (2.73 Kelvin)^4,
  Gram / (Centi Meter)^3]
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$$\frac{1.16897\times 10^{-34}\ \text{Gram}}{\text{Centi}^3\ \text{Meter}^3}$$

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% / %%
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0.0000119944

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1.2*^-5 / .27
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0.0000444444

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1 / %
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22 500.

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1.2*^-5 / .043
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0.00027907

```
1 / %
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3583.33

Calculation of the sound speed

- The composition of the gas is ordinary matter and photons.
- Q: Does matter or radiation provide more pressure? The answer is detailed, but it is based on a principle. What principle is the basis for calculating the answer?
  - Pressure $P = n\, p_x\, v_x$, where n is number density, p and v are momentum and speed. For matter, $P = n\, m\, v_x^2 = \frac{1}{3}\, n\, m\, v^2$. For radiation, $P = \frac{1}{3}\, n\, E$.
  - Equipartition: In thermal equilibrium, the energy of each particle is the same. (More precisely, the energy of each degree of freedom is $\frac{1}{2}\, k\, T$. In QM, degrees of freedom may be frozen with 0 energy.)
  - Matter: $P = \frac{2}{3}\, n\, \frac{3}{2}\, k\, T = n\, k\, T$.
    $P = u$, where u is the energy density.
  - Radiation: $P \approx \frac{1}{3}\, n\, \frac{1}{2}\, k\, T$. More accurately, $P = 0.90\, n\, k\, T$.
    $P = \frac{1}{3}\, u$.
  - There are 10^9 photons for every baryon.
  - Radiation also dominates the energy density. (Not the case if the rest mass density is added in.)

Consider a box of gas with a fixed number of particles. The box expands or shrinks because of the sound wave.

1. $d\,U = d\,Q - P\,d\,V = -P\,d\,V$

Because there is no heat flow, $d\,Q = 0$.

$$d\,U = -P\,d\,V.$$

Recall $u = a_B\, T^4$ and $P = \frac{1}{3}\, a_B\, T^4$

$$d\,(u\,V) = V\,d\,u + u\,d\,V = -P\,d\,V$$
$$d\,u = -(u + P)\,d\,V / V$$
$$4\,T^3\,d\,T = -\left(T^4 + \frac{1}{3}\,T^4\right)\mathrm{dV} / V$$
$$3\,d\,T / T = -d\,V / V$$

2. $d\,P = \frac{4}{3}\, a_B\, T^3\, d\,T$.

3. $d\,\rho = d\,\rho_b + d\,\rho_r$

$d\,\rho_b = -\rho_b\, d\,V / V$, since mass of the baryons in the box $(\rho_b\, V)$ is unchanged.

$$d\,\rho_b = 3\,\rho_b\, d\,T / T$$

$$d\,\rho_r = 4\, a_B\, T^3\, d\,T$$

4. Gather all:

$$v_s^2 = \frac{dP}{d\rho} = \left(\frac{4}{3}\, a_B\, T^3\, d\,T\right)\left(3\,\rho_b\, d\,T / T + 4\, a_B\, T^3\, d\,T\right)^{-1}$$
$$= \left(3 + \frac{9}{4}\,\frac{\rho_b}{\rho_r}\right)^{-1}$$
$$\boxed{v_s = [3\,(1 + R)]^{-1/2}},$$

where

$$R = \frac{3}{4}\,\frac{\rho_b}{\rho_r}$$

Q: If $R \ll 1$, how fast do sound waves travel?

Q: Why do baryons slow the speed of sound? Recall $v_s = \left(\frac{d\,P}{d\,\rho}\right)^{1/2}$.

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Number density of photons:

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Integrate[x^2 / (ⅇ^x - 1), {x, 0, ∞}]
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2 Zeta[3]

Energy density:

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Integrate[x^3 / (ⅇ^x - 1), {x, 0, ∞}]
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$$\frac{\pi^4}{15}$$

Average energy:

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% / %% // N
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2.70118

$\frac{1}{3}\langle E\rangle$ is

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% / 3
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0.900393

Calculation of the horizon, step 1

How far does a sound wave travels from the big bang (t=0) to the time of recombination?

Let

a_L	be the expansion parameter at last scattering (recombination)
a_E	be the expansion parameter at epoch when $\rho_r = \rho_b$.

The distance of the horizon is

$$d = \int v_s \, d\, t.$$

$v_s \, d\, t$ is how far the sound wave moves. As the wave is moving, the starting point is moving too. Calculation is wrong.

Better posed: What is the comoving coordinate r of a sound wave that travels from the big bang (t=0) to the epoch of recombination?

$$v_s \, d\, t = a \, d\, r$$

$$r = \int_0^{t_L} v_s \, a^{-1} \, d\, t.$$

Q: Given r, how do you get the distance of the horizon?

Calculation of the horizon, step 2

How far does a sound wave travels from the big bang to the epoch of recombination?
Let

a_L be the expansion parameter at last scattering (recombination)
a_E be the expansion parameter at epoch when $\rho_r = \rho_b$.

The distance of the horizon is

$$d = \int v_s\, d\, t.$$

$v_s\, d\, t$ is how far the sound wave moves. As the wave is moving, the starting point is moving too. Calculation is wrong.

Better posed: What is the comoving coordinate r of a sound wave that travels from the big bang to the epoch of recombination?

$$v_s\, d\, t = a\, d\, r$$

$$r = \int_0^{t_L} v_s\, a^{-1}\, d\, t.$$

Q: Given r, how do you get the distance of the horizon? $d = a_L\, r$

$$d = a_L \int_0^{t_L} v_s\, a^{-1}\, d\, t.$$

The sound speed $v_s = [3\,(1+R)]^{-1/2}$ depends on $R = \frac{3}{4}\,\frac{\rho_b}{\rho_r} = \frac{3}{4}\left(\frac{a}{a_E}\right)$.

- **To gain understanding, consider this simplified case: $R \ll 1$.**

Then

$$d = a_L\, v_s \int_0^{t_L} a^{-1}\, d\, t$$

Use Friedman's equation

$$\left(\frac{d\, a}{H_0\, d\, t}\right)^2 = \left(\Omega_{k0} + \Omega_{m0}\, a^{-1} + \Omega_{r0}\, a^{-2} + \Omega_{v0}\, a^2\right) \to \Omega_{r0}\, a^{-2}$$

$$d = a_L\, v_s \int_0^{t_L} a^{-1}\, d\, t$$

$$= \; H_0^{-1}\, a_L\, v_s\, \Omega_{r0}^{-1/2} \int_0^{a_L} d\, a$$

$$= \; H_0^{-1}\, a_L^2\, v_s\, \Omega_{r0}^{-1/2}$$

Apply F's eqn at a_L

$$H(a_L) = H_0\, \Omega_{r0}^{1/2}\, a_L^{-2}$$

to get the transparent result

$$d = H^{-1}(a_L)\, v_s$$

Q: Interpret the formula for d.
A dense region produces a sound wave that goes in all directions to cover a length $2\,d$.
The angle subtended is

$$\theta = \frac{2\,d}{r\, a_L}$$

$$= \; \frac{2\, H^{-1}(a_L)\, v_s}{a_L\, r(a_L)}$$

Q: Interpret the formula for θ.

- **Results for best cosmological values**

$$d = a_L \int_0^{t_L} a^{-1}\, v_s(a)\, d\, t$$

Change $d\, t = H^{-1}\, a^{-1}\, d\, a$, and integrate to get (Weinberg 2008, Cosmology, p. 145)

$$d = 2\, H_0^{-1}\, a_L^{-3/2} (3\, R_L\, \Omega_{\text{m0}})^{-1/2} \ln\left\{\left[(1 + R_L)^{1/2} + (R_E + R_L)^{1/2}\right] / \left(1 + \sqrt{R_E}\right)\right\}$$

For $\Omega_{\text{m0}} = 0.26$, $\Omega_{\text{v0}} = 0.74$, $\Omega_{\text{b0}} = 0.043$,

$$R_L = 0.62$$

$$R_E = 0.21$$

$$d = 1.16\, H_0^{-1}\, a_L^{3/2}$$

$$\theta = \frac{2\, d}{r(a_L)\, a_L} = 1/48 = 1.2\,°$$