Tensors—18 March 2010

- Outline for the next month
  - Equivalence Principle shows Einstein how to incorporate gravity and special relativity. The happiest thought. (Done)
  - Experimental tests of the equivalence principle. (Done)
  - Mathematics: scalars, vectors, tensors
  - Following Einstein's path to E's field equations
  - Mathematics of curvature
  - Bianchi's identity
  - Discovery of E's field equation. November 1915.

- Reading for today
  - Hartle §20
  - Weinberg §2.1 and §4.1, 4.2, 4.3, 4.6)
Lorentz transformation

Lessons from special relativity:
1) Laws of physics must be written in scalars, vectors, or tensors. They cannot be parts of a vector.
2) Scalars, vectors, and tensors are defined by their transformation properties.

Consider the space-time coordinates \(x^\mu\). In a different coordinate system, the coordinates are \(x'^\mu\). The Lorentz transformation is
\[
x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu.
\]
The Lorentz transformation satisfies
\[
\eta_{\gamma\delta} \Lambda^\gamma_\alpha \Lambda^\delta_\beta \eta_{\alpha\beta} = \eta_{\gamma\delta},
\]
where
\[
\eta = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
Q: Name a transformation.

What is special about a Lorentz transformation? The length of \(dx^\mu\) is defined to be
\[
-\eta_{\mu\nu} dx^\mu dx^\nu.
\]
In a different frame,
\[
-\eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\gamma\delta} \Lambda^\mu_\gamma \Lambda^\nu_\delta dx^\gamma dx^\delta
\]
\[
= \eta_{\gamma\delta} \Lambda^\mu_\gamma \Lambda^\nu_\delta \eta_{\gamma\delta} dx^\gamma dx^\delta
\]
\[
= \eta_{\mu\nu} dx^\mu dx^\nu
\]
\[
= -dt^2 + d\vec{x}^2
\]
Because the laws of physics are not invariant under all Lorentz transformations, let us impose
\[
\Lambda^0_0 \geq 1
\]
and
\[
\det \Lambda = +1.
\]
Q: What transformations do these restrictions forbid?

These transformations are called proper, inhomogeneous Lorentz transformations.
Proper means with the restrictions.
Inhomogeneous means \(a^\mu \neq 0\).
Equivalence Principle—General Covariance

Principle of General Covariance: A statement of physics is true if
1) it is true in the absence of gravity, and
2) it preserves its form under a general coordinate transformation.

Example: The equation of motion
\[
\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0
\]
preserves its form under a general coordinate transformation. In the absence of gravity, it is
\[
\frac{du^\alpha}{d\tau} = 0,
\]
which is true.

Equivalence Principle: Suppose a case where gravity is present. There exists a frame in which gravity is absent. Write the law of physics in this frame. Then transform back to the original frame.

Restated: To eliminate gravity, transform to a gravity-free frame. To account for gravity, transform back to the original frame.

Q: Are Lorentz transformations and a general coordinate transformation the same?
Q: What is the purpose of a Lorentz transformation? What is the purpose of a general coordinate transformation?
Scalars, vectors, and tensors

Scalars do not change under a general coordinate transformation.

A contravariant vector transforms in the same way as $d x^a$.

$$dx'^a = dx^c \frac{\partial x'^a}{\partial x^c}$$

$$A'^a = A^n \frac{\partial x'^a}{\partial x^n}$$

A covariant vector transforms as

$$A'_\mu = A_\nu \frac{\partial x^\nu}{\partial x'^\mu}$$

Q: Is the derivative of a scalar $\frac{\partial b}{\partial c}$ a contravariant or covariant vector?

A tensor $T^{\mu \nu}$ transforms like $A_\mu B^\nu$. A tensor $T_{\mu \nu}$ transforms like $A_\mu B_\nu$, etc.

Prove that the metric tensor is a tensor.

1) There is a frame in which the metric tensor is $\eta_{ab}$. The position vector is $x^a$.

Q: Reason?

2) For a general coordinate frame $x^a$,

$$g_{\mu \nu} \equiv \eta_{ab} \frac{\partial x^a}{\partial x'^\mu} \frac{\partial x^b}{\partial x'^\nu}$$

3) In a different coordinate frame $x'^a$,

$$g'_{\mu \nu} \equiv \eta_{ab} \frac{\partial x'^a}{\partial x^\mu} \frac{\partial x'^b}{\partial x^\nu}$$

$$= \eta_{ab} \frac{\partial x^a}{\partial x'^\mu} \frac{\partial x^b}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\mu} \frac{\partial x'^\nu}{\partial x^\nu}$$

Q: Reason?

4) $g'_{\mu \nu} = \eta_{ab} \frac{\partial x^a}{\partial x'^\mu} \frac{\partial x^a}{\partial x'^\nu}$

$$= \eta_{ab} \frac{\partial x^a}{\partial x'^\mu} \frac{\partial x^a}{\partial x'^\nu}$$

Q: Reason?

Q: Is $g_{\mu \nu}$ a covariant or contravariant or mixed tensor?
Tensor algebra

Linear combination:
If \( a \) and \( b \) are scalars, and \( A^{\mu \nu} \) and \( B^{\mu \nu} \) are tensors, then
\[ a \, A^{\mu \nu} + b \, B^{\mu \nu} \]
is a tensor.
Q: How do you prove this?

Direct product
If \( A^{\alpha \beta} \) and \( B^{\mu \nu} \) are tensors, then
\[ A^{\alpha \beta} \, B^{\mu \nu} \]
is a tensor.

Contraction
If \( A^{\alpha \beta} \) and \( B_{\mu}^{\nu} \) are tensors, then
\[ T^{\nu}_{\alpha} = A^{\alpha \beta} \, B^{\mu}_{\beta} \]
is a tensor.
Special but important case: The proper time \( d\tau \) of a time-like vector \( dx^\alpha \) is
\[ d\tau^2 = -dx_\mu \, dx^\mu. \]
Covariant derivative of a contravariant vector

How do you take derivatives of tensors?

We already found that the equation of motion is
\[ \frac{du^a}{d\tau} + \Gamma^a_{\beta \gamma} u^\beta u^\gamma = 0. \]

The terms \( \frac{du^a}{d\tau} \) and \( \Gamma^a_{\beta \gamma} u^\beta u^\gamma \) are not tensors. Proof: \( \Gamma^a_{\beta \gamma} u^\beta u^\gamma \) is zero in a gravity-free frame. If it were a tensor, it must be zero in all frames.

We derived the equation of motion by differentiating the 4-velocity.
Rewrite
\[ \frac{du^a}{d\tau} = \frac{d \vec{x}^\xi}{d\tau} \frac{\partial u^a}{\partial \vec{x}^\xi} = u^\beta \frac{\partial u^a}{\partial \vec{x}^\xi} \]
and insert to get
\[ u^\beta \left( \frac{\partial u^a}{\partial \vec{x}^\xi} + \Gamma^a_{\beta \gamma} u^\gamma \right) = 0. \]
This says: In the parenthesis is the change in \( u^a \) in the \( \vec{x}^\xi \) direction. Contracting it (taking the dot product) with \( u^\beta \) results in 0.

Contraction is a tensor operation. \( \frac{\partial u^a}{\partial \vec{x}^\xi} + \Gamma^a_{\beta \gamma} u^\gamma \) is a tensor.
For any contravariant vector \( A^\alpha \),
\[ \nabla_\beta A^\alpha = \frac{\partial A^\alpha}{\partial \vec{x}^\xi} + \Gamma^\alpha_{\beta \gamma} A^\gamma \]
is a tensor. This is called the covariant derivative. Another notation:
\[ A^\alpha_{\beta} = A^\alpha_{\beta} + \Gamma^\alpha_{\beta \gamma} A^\gamma \]
Q: Is \( A^\alpha_{\beta} \equiv \nabla_\beta A^\alpha \) covariant or contravariant in the index \( \beta \)?

Example: For 2-dimensional polar coordinates, the metric is
\[ ds^2 = dr^2 + r^2 d\theta^2 \]
The non-zero Christoffel symbols are (8.17)
\[ \Gamma^\theta_{\phi \theta} = -r \]
\[ \Gamma^\phi_{\theta \phi} = \Gamma^\phi_{\phi \theta} = 1/r. \]

\[ A^\alpha_{\phi} = A^\alpha_{\phi} \]
\[ A^\alpha_{\theta} = A^\alpha_{\theta} - r A^\phi \]
The covariant derivative of the \( r \) component in the \( r \) direction is the regular derivative. If a vector field is constant, then \( A^r_r = 0 \).

The covariant derivative of the \( r \) component in the \( \theta \) direction is the regular derivative plus another term. Even if a vector field is constant, \( A^r_{\theta} \neq 0 \). The \( \Gamma \) term accounts for the change in the coordinates.

The idea of a covariant derivative of a vector field \( A \) in the direction \( a \). Is this a good definition?

\[
\nabla_a A^a = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ A(x + \epsilon a) - A(x) \right] \]

However, the components of \( A(x + \epsilon a) \) may be different even if the vector is the same, because the coordinates are changing. We must move \( A(x + \epsilon a) \) back to \( x \) before comparing. Moving is called parallel transporting. This is what the \( \Gamma \) term does.

\[
\nabla_a A^a = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \text{parallel transport} \left( A(x + \epsilon a) \right) - A(x) \right]
\]

Q: Simplicio: Covariant derivatives are irrelevant. I want to know about gravity. In what way is Simplicio mistaken?