Tensors—18 March 2010

- Outline for the next month
  - Equivalence Principle shows Einstein how to incorporate gravity and special relativity. The happiest thought. (Done)
  - Experimental tests of the equivalence principle. (Done)
  - Mathematics: scalars, vectors, tensors
  - Following Einstein's path to E's field equations
  - Mathematics of curvature
  - Bianchi's identity
  - Discovery of E's field equation. November 1915.
- Reading for today
  - Hartle §20
  - Weinberg §2.1 and §4.1, 4.2, 4.3, 4.6)

Lorentz transformation

Lessons from special relativity:
1) Laws of physics must be written a scalers, vectors, or tensors. They cannot be parts of a vector.
2) Scalars, vectors, and tensors are defined by their transformation properties.

Consider the space-time coordinates x^μ. In a different coordinate system, the coordinates are x'^μ. The Lorentz transformation is

$$x'^\alpha = \Lambda^\alpha{}_\beta\, x^\beta + a^\alpha.$$

The Lorentz transformation satisfies

$$\Lambda^\alpha{}_\gamma\, \Lambda^\beta{}_\delta\, \eta_{\alpha\beta} = \eta_{\gamma\delta},$$

where

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Q: Name a transformation.

What is special about a Lorentz transformation? The length of dx^α is defined to be

$$-\mathrm{dt}^2 + \overrightarrow{\mathrm{dx}}^2 = \eta_{\alpha\beta}\, \mathrm{dx}^\alpha\, \mathrm{dx}^\beta.$$

In a different frame,

$$\begin{aligned} -\mathrm{dt}'^2 + \overrightarrow{\mathrm{dx}'}^2 &= \eta_{\alpha\gamma}\, \mathrm{dx}'^\alpha\, \mathrm{dx}'^\gamma. \\ &= \eta_{\alpha\gamma}\, \Lambda^\alpha{}_\beta\, \mathrm{dx}^\beta\, \Lambda^\gamma{}_\delta\, \mathrm{dx}^\delta \\ &= \eta_{\beta\delta}\, \mathrm{dx}^\beta\, \mathrm{dx}^\delta \\ &= -\mathrm{dt}^2 + \overrightarrow{\mathrm{dx}} \end{aligned}$$

Because the laws of physics are not invarient under all Lorentz transformations, let us impose

$$\Lambda^0{}_0 \geq 1$$

and

$$\det \Lambda = +1.$$

Q: What transformations do these restrictions forbid?

These transformations are called proper, inhomogeneous Lorentz transformations.
Proper means with the restrictions.
Inhomogeneous means $a^\alpha \neq 0$.

Equivalence Principle—General Covariance

Principle of General Covariance: A statement of physics is true if
1) it is true in the absence of gravity, and
2) it preserves its form under a general coordinate transformation.

Example: The equation of motion

$$\frac{du^\alpha}{d\tau}+\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma = 0$$

preserves its form under a general coordinate transformation. In the absence of gravity, it is

$$\frac{du^\alpha}{d\tau} = 0,$$

which is true.

Equivalence Principle: Suppose a case where gravity is present. There exists a frame in which gravity is absent. Write the law of physics in this frame. Then transform back to the original frame.

Restated: To eliminate gravity, transform to a gravity-free frame. To account for gravity, transform back to the original frame.

Q: Are Lorentz transformations and a general coordinate transformation the same?
Q: What is the purpose of a Lorentz transformation? What is the purpose of a general coordinate transformation?

Scalars, vectors, and tensors

Scalars do not change under a general coordinate transformation.

A contravariant vector transforms in the same way as $d\,x^\alpha$.

$$\mathrm{dx}'^\mu = \mathrm{dx}^\nu \frac{\partial x'^\mu}{\partial x^\nu}$$

$$A'^\mu = A^\nu \frac{\partial x'^\mu}{\partial x^\nu}$$

A covariant vector transforms as

$$A'_\mu = A_\nu \frac{\partial x^\mu}{\partial x'^\nu}$$

Q: Is the derivative of a scalar $\frac{\partial \phi}{\partial x^\mu}$ a contravariant or covariant vector?

A tensor $T_\mu{}^\nu$ transforms like $A_\mu\,B^\nu$. A tensor $T_{\mu\nu}$ transforms like $A_\mu\,B_\nu$, etc.

Prove that the metric tensor is a tensor.

1) There is a frame in which the metric tensor is $\eta_{\alpha\beta}$. The position vector is ξ^α.

Q: Reason?

2) For a general coordinate frame x^α,

$$g_{\mu\nu} \equiv \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}.$$

3) In a different coordinate frame x'^α,

$$g'_{\mu\nu} \equiv \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x'^\mu} \frac{\partial \xi^\beta}{\partial x'^\nu}$$

$$= \quad \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial \xi^\beta}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\nu}$$

Q: Reason?

4) $$g'_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial \xi^\beta}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\nu}$$

$$= \quad g_{\sigma\rho} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu}$$

Q: Reason?

Q: Is $g_{\mu\nu}$ a covariant or contravariant or mixed tensor?

Tensor algebra

Linear combination:
If a and b are scalars, and $A^{\mu\nu}$ and $B^{\mu\nu}$ are tensors, then

$$a A^{\mu\nu} + b B^{\mu\nu}$$

is a tensor.
Q: How do you prove this?

Direct product
If $A^{\alpha\beta}$ and $B^{\mu\nu}$ are tensors, then

$$A^{\alpha\beta} B^{\mu\nu}$$

is a tensor.

Contraction
If $A^{\alpha\beta}$ and $B_{\mu}{}^{\nu}$ are tensors, then

$$T^{\alpha\nu} \equiv A^{\alpha\beta} B_{\beta}{}^{\nu}$$

is a tensor.
Special but important case: The proper time $d\tau$ of a time-like vector dx^{α} is

$$d\tau^2 = -dx_{\alpha}\, dx^{\alpha}.$$

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Covariant derivative of a contravariant vector

How do you take derivatives of tensors?

We already found that the equation of motion is

$$\frac{du^\alpha}{d\tau}+\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma = 0.$$

The terms $\frac{du^\alpha}{d\tau}$ and $\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma$ are not tensors. Proof: $\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma$ is zero in a gravity-free frame. If it were a tensor, it must be zero in all frames.

We derived the equation of motion by differentiating the 4-velocity.
Rewrite

$$\frac{du^\alpha}{d\tau}= \frac{dx^\beta}{d\tau}\frac{\partial u^\alpha}{\partial x^\beta}= u^\beta \frac{\partial u^\alpha}{\partial x^\beta}$$

and insert to get

$$u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, u^\gamma\right) = 0.$$

This says: In the parenthesis is the change in u^α in the x^β direction. Contracting it (taking the dot product) with u^β results in 0.

Contraction is a tensor operation. $\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, u^\gamma$ is a tensor.
For any contravarient vector A^α,

$$\boxed{\nabla_\beta A^\alpha = \frac{\partial A^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, A^\gamma}$$

is a tensor. This is called the covariant derivative. Another notation:

$$\boxed{A^\alpha{}_{;\beta} = A^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma}\, A^\gamma}$$

Q: Is $A^\alpha{}_{;\beta} \equiv \nabla_\beta A^\alpha$ covariant or contravarient in the index β?

Example: For 2-dimensional polar coordinates, the metric is

$$\mathrm{ds}^2 = \mathrm{dr}^2 + r^2\, \mathrm{d}\theta^2$$

The non-zero Christoffel symbols are (8.17)

$$\Gamma^r_{\theta\theta} = -r$$

$$\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = 1/r.$$

$$A^r{}_{;r} = A^r{}_{,r}$$

$$A^r{}_{;\theta} = A^r{}_{,\theta} - r\, A^\theta$$

$A^{\theta}{}_{;r} = A^{\theta}{}_{,r} + 1/r\, A^{\theta}$

$A^{\theta}{}_{;\theta} = A^{\theta}{}_{,\theta} + 1/r\, A^{r}$

The covariant derivative of the r component in the r direction is the regular derivative. If a vector field is constant, then $A^{r}{}_{;r} = 0$.

The covariant derivative of the r component in the θ direction is the regular derivative plus another term. Even if a vector field is constant, $A^{r}{}_{;\theta} \neq 0$. The Γ term accounts for the change in the coordinates.

The idea of a covariant derivative of a vector field A in the direction a. Is this a good definition?

$\nabla_a A^{\alpha} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [A(x + \epsilon\, a) - A(x)]$???

However, the components of $A(x + \epsilon\, a)$ may be different even if the vector is the same, because the coordinates are changing. We must move $A(x + \epsilon\, a)$ back to x before comparing. Moving is called parallel transporting. This is what the Γ term does.

$\nabla_a A^{\alpha} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{\text{parallel transport}[A(x + \epsilon\, a)] - A(x)\}$

Q: Simplicio: Covariant derivatives are irrelavant. I want to know about gravity. In what way is Simplicio mistaken?