Riemann-Christoffel curvature tensor—23 Mar 2010

- Reading: Weinberg Gravitation & Cosmology, §6 and Hartle §21.3
- Outline
  - Finish covariant derivatives
  - Riemann-Christoffel curvature tensor
  - Bianchi identity
Covariant derivative of a contravariant vector

How do you take derivatives of tensors?

We already found that the equation of motion is
\[ \frac{du^a}{d\tau} + \Gamma^a_{\beta \gamma} u^\beta u^\gamma = 0. \]

The terms \( \frac{du^a}{d\tau} \) and \( \Gamma^a_{\beta \gamma} u^\beta u^\gamma \) are not tensors. Proof: \( \Gamma^a_{\beta \gamma} u^\beta u^\gamma \) is zero in a gravity-free frame. If it were a tensor, it must be zero in all frames.

We derived the equation of motion by differentiating the 4-velocity. Rewrite
\[ \frac{du^a}{d\tau} = \frac{dx^a}{d\tau} \frac{du^a}{dx^b} = u_b^{\beta} \frac{du^\beta}{dx^a} \]

and insert to get
\[ u_\beta \left( \frac{\partial u^a}{\partial x^b} + \Gamma^a_{\beta \gamma} u^\gamma \right) = 0. \]

This says: In the parenthesis is the change in \( u^a \) in the \( x^\beta \) direction. Contracting it (taking the dot product) with \( u_\beta \) results in 0.

Contraction is a tensor operation. \( \frac{\partial u^a}{\partial x^b} + \Gamma^a_{\beta \gamma} u^\gamma \) is a tensor.

For any contravariant vector \( A^a \),
\[ \nabla_\beta A^a = \frac{\partial A^a}{\partial x^\beta} + \Gamma^a_{\beta \gamma} A^\gamma \]
is a tensor. This is called the covariant derivative. Another notation:
\[ A^{a}_{;\beta} = A^{a}_{\beta} + \Gamma^a_{\beta \gamma} A^\gamma \]

Q: Is \( A^{a}_{;\beta} \equiv \nabla_\beta A^a \) covariant or contravariant in the index \( \beta \)?

Example: For 2-dimensional polar coordinates, the metric is
\[ ds^2 = dr^2 + r^2 \, d\theta^2 \]
The non-zero Christoffel symbols are (8.17)
\[ \Gamma^r_{rr} = -r \]
\[ \Gamma^\theta_{\theta r} = \Gamma^\theta_{r \theta} = 1/r. \]
\[ A^{r'}_{;r} = A^{r'}_{r} \]
The covariant derivative of the \( r \) component in the \( r \) direction is the regular derivative. If a vector field is constant, then \( A'_r = 0 \).

The covariant derivative of the \( r \) component in the \( \theta \) direction is the regular derivative plus another term. Even if a vector field is constant, \( A'_\theta \neq 0 \). The \( \Gamma \) term accounts for the change in the coordinates.

The idea of a covariant derivative of a vector field \( \mathbf{A} \) in the direction \( \mathbf{a} \). Is this a good definition?

\[
\nabla_a \mathbf{A} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\mathbf{A}(x + \epsilon \mathbf{a}) - \mathbf{A}(x)]
\]

However, the components of \( \mathbf{A}(x + \epsilon \mathbf{a}) \) may be different even if the vector is the same, because the coordinates are changing. We must move \( \mathbf{A}(x + \epsilon \mathbf{a}) \) back to \( x \) before comparing. Moving is called parallel transporting. This is what the \( \Gamma \) term does.

\[
\nabla_a \mathbf{A} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [\text{parallel transport}[\mathbf{A}(x + \epsilon \mathbf{a})] - \mathbf{A}(x)]
\]

Q: Simplicio: Covariant derivatives are irrelevant. I want to know about gravity. In what way is Simplicio mistaken?
**Covariant derivative of a tensor**

A contravariant tensor is $A^\mu B^\nu$. The covariant derivative of it is

$$
(A^\mu B^\nu)_{\alpha\beta} = A^\mu(B^\nu)_{\alpha\beta} + (A^\mu)_{\alpha\beta} B^\nu
$$

$$
= A^\mu(B^\nu,_{\alpha} + \Gamma^\nu_{\beta\alpha} B^\beta) + B^\nu(A^\mu,_{\alpha} + \Gamma^\mu_{\beta\alpha} A^\beta)
$$

$$
= (A^\mu B^\nu)_{\alpha} + A^\mu B^\beta \Gamma^\nu_{\beta\alpha} + B^\nu A^\beta \Gamma^\mu_{\beta\alpha}
$$

Therefore the covariant derivative of a contravariant tensor is

$$
T^{\mu\nu}_{\alpha\beta} = T^{\mu\nu}_{\alpha\beta} + T^{\mu\beta} \Gamma^\nu_{\beta\alpha} + T^{\nu\beta} \Gamma^\mu_{\beta\alpha}
$$

There is one Christoffel symbol for each upper index.

The covariant derivative of a covariant vector is

$$
A_{\alpha;\beta} = A_{\alpha;\beta} - \Gamma^\gamma_{\alpha\beta} A_{\gamma}
$$

Proof: Find the covariant derivative of $A_{\alpha;\alpha}$.

The covariant derivative of a mixed tensor: Put in $+\Gamma$ for each upper index and $-\Gamma$ for each lower index.
How to measure curvature

Q: In what object is gravity encoded? What does the Equivalence Principle say?

Q: Can you measure curvature by looking at a point?
How to measure curvature

Q: In what object is gravity encoded? What does the Equivalence Principle say? Gravity is encoded in a general coordinate transformation.

Q: Can you measure curvature by looking at a point? No. The equivalence principle says that gravity can be removed in a small region of space-time by a coordinate transformation. You must explore a region that is not small.

Q: How to detect curvature of the Earth's surface.
Carry a vector, which points east, from the north pole to the equator.

Consider a vector field $A_x$. Move from point P to Q to R. Move from P to S to R. Compare.

The change in $A$ in going from P to Q is

$$dA_{PQ} = \left( \frac{\partial A_x}{\partial x^a} \right) a^a$$

Q: Why is this not a tensor equation?

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In[54]:= fig[] := Module[{p, q, r, s, x},
  p = {0, 0}; q = {1, 0}; s = {.2, 1}; r = {1.2, 1};
  x = {p, q, r, s};
  ListPlot[x, PlotStyle -> PointSize -> Large],
  Epilog -> {Text["P", p, {-3, -1}], Text["Q", q, {2, -1}], Text["S", s, {-2, 1}],
    Text["R", r, {3, 1}], Text["a", Mean@{x[[1]], x[[2]]}, {0, -1}],
    Text["b", Mean@{x[[1]], x[[4]]}, {-2.5, 0}], Arrow[{x[[1]], x[[2]]}], Arrow[{x[[2]], x[[3]]}],
    Arrow[{x[[1]], x[[4]]}], Arrow[{x[[4]], x[[3]]}]}, bs, Axes -> None, ImageSize -> 160]
]
In[55]:= fig[]
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Out[55]=
How to measure curvature

Consider a vector field $A_{\gamma}$. Move from point P to Q to R. Move from P to S to R. Compare.

The change in $A_{\gamma}$ in going from P to Q is

$$dA_{\gamma}^{PQ} = \left(\frac{\partial A_{\gamma}}{\partial x^a}\right) dx^a$$

Q: Why is this not a tensor equation? The derivative of a vector field is not a tensor. Use the covariant derivative

$$\nabla_{\alpha} A_{\gamma} = \frac{\partial A_{\gamma}}{\partial x^\alpha} - \Gamma^\gamma_{\alpha\beta} A_{\beta}$$

This is a tensor equation:

$$dA_{\gamma}^{PQ} = \nabla_{\alpha} A_{\gamma} \, dx^\alpha$$

The change in $A$ in going P→Q→R is

$$dA_{\gamma}^{PQR} = \nabla_{\beta} (\nabla_{\alpha} A_{\gamma}) \, dx^\alpha \, dx^\beta$$

The change in $A$ in going P→S→R is

$$dA_{\gamma}^{PSR} = \nabla_{\alpha} (\nabla_{\beta} A_{\gamma}) \, dx^\alpha \, dx^\beta$$

The change in a round trip P→Q→R→S→P is

$$dA_{\gamma}^{PQ} - dA_{\gamma}^{PSR} = \left[\nabla_{\beta} (\nabla_{\alpha} A_{\gamma}) - \nabla_{\alpha} (\nabla_{\beta} A_{\gamma})\right] \, dx^\alpha \, dx^\beta$$

Q: In MA1, I learned that $\frac{\partial^2}{\partial y \partial x} = -\frac{\partial^2}{\partial x \partial y}$. Why doesn't the quantity in brackets $[] = 0$?
How to measure curvature

Consider a vector field $A_x$. Move from point P to Q to R. Move from P to S to R. Compare.

The change in $A_x$ in going from P to Q is

$$dA_{PQ} = \left( \frac{\partial A_x}{\partial x^a} \right) dx^a$$

Q: Why is this not a tensor equation? The derivative of a vector field is not a tensor. Use the covariant derivative

$$\nabla_a A_g = \frac{\partial A_x}{\partial x^a} - \Gamma^r_{ya} A_r$$

The change in a round trip $P \rightarrow Q \rightarrow R \rightarrow S \rightarrow P$ is

$$dA_{PQ} - dA_{PSR} = [\nabla_\beta (\nabla_a A_g) + \nabla_a (\nabla_\beta A_g)] dx^a b^\beta$$

Q: In MA1, I learned that $\frac{\partial^2}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial y}$. Why doesn't the quantity in brackets $[] = 0$? The parts involving partial derivatives of the vector $A_x$ is 0. The remaining parts involve the Christoffel symbol times $A$. Therefore, the nonzero part can be written as

$$dA_{PQ} - dA_{PSR} = -A_x R^r_{yab} a^a b^\beta.$$  

What does this say?

Q: In a round trip, a vector field $A_x$ changes by the contraction of what?
Riemann-Christoffel curvature tensor

Consider a vector field $A_y$. Move from point P to Q to R. Move from P to S to R. Compare.

![Diagram of vectors P, Q, R, S, a, b]

The change in $A_y$ in going from P to Q is

$$d A_y^{PQ} = \left( \frac{\partial A_y}{\partial x} \right) d^\alpha$$

Q: Why is this not a tensor equation? The derivative of a vector field is not a tensor. Use the covariant derivative

$$\nabla_a A_y = \frac{\partial A_y}{\partial x^a} - \Gamma^\gamma_{ya} A_\gamma$$

The change in a round trip P→Q→R→S→P is

$$d A_y^{PQR} - d A_y^{PSR} = \left[ \nabla_\beta \left( \nabla_a A_y \right) - \nabla_a \left( \nabla_\beta A_y \right) \right] d^\alpha b^\beta$$

Q: In MA1, I learned that $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$. Why doesn't the quantity in brackets $[] = 0$? The parts involving partial derivatives of the vector $A_y$ is 0. The remaining parts involve the Christoffel symbol times $A$. Therefore, the nonzero part can be written as

$$d A_y^{PQR} - d A_y^{PSR} = -A_\sigma R^\gamma_{\sigma a} a^\alpha b^\beta$$

What does this say?

Q: In a round trip, a vector field $A_y$ changes by the contraction of $A$, a tensor $R$, the position change $a$, and the position change $b$. The tensor $R^\gamma_{\sigma a}$ is called the Riemann-Cristoffel curvature tensor.

Q: If I swap $\alpha$ and $\beta$, is $R$ the same? $R^\gamma_{\sigma a} = R^\gamma_{\sigma b}$? What are the last two indices for?

Calculating $R^\gamma_{\sigma a}$:

$$\nabla_\beta \left( \nabla_a A_y \right) = \nabla_\beta \left( \frac{\partial A_y}{\partial x^a} - \Gamma^\gamma_{ya} A_\gamma \right) = \frac{\partial^2 A_y}{\partial x^a \partial x^\beta} - A_\sigma \frac{\partial}{\partial x^a} \Gamma^\sigma_{\gamma a} - \Gamma^\sigma_{y a} \frac{\partial A_\gamma}{\partial x^a} - \Gamma^\gamma_{\sigma b} \frac{\partial A_\beta}{\partial x^a}$$

We can ignore the partial derivatives of $A$, because in the end only the terms in $A$ survive. It is possible to show that

$$R^\gamma_{\sigma a} = \frac{\partial}{\partial x^\sigma} \Gamma^\gamma_{y b} - \frac{\partial}{\partial x^b} \Gamma^\gamma_{y a} + \Gamma^\sigma_{a e} \Gamma^\gamma_{e b} - \Gamma^\gamma_{b e} \Gamma^\sigma_{e a}$$
Ricci tensor and curvature scalar, symmetry

The Ricci tensor is a contraction of the Riemann-Christoffel tensor
\[ R_{\gamma\delta} \equiv R^{\alpha}_{\gamma\alpha\delta}. \]

The curvature scalar is the contraction of the Ricci tensor
\[ R = g^{\delta\gamma} R_{\gamma\delta}. \]

Symmetry properties of the Riemann-Christoffel tensor \( R_{\alpha\beta\gamma\delta} \equiv g_{\alpha\tau} R^{\tau}_{\beta\gamma\delta} \)

1) Symmetry is swapping the first and second pair
\[ R_{\alpha\beta\gamma\delta} = R_{\beta\alpha\gamma\delta}. \]

2) Antisymmetry in swapping first pair or second pair
\[ R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\delta\beta\gamma}. \]

3) Cyclicity in the last three indices.
\[ R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0. \]
Example: Surface of a 2-d sphere

The metric is
\[ ds^2 = a^2(\theta^2 + \sin^2 \theta \, d\phi^2). \]

The nonzero parts of the Christoffel symbol are
\[ \Gamma^\phi_{\theta\phi} = -\sin \theta \cos \theta \]
\[ \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\theta} = -\sin \theta \cos \theta \]

The Riemann-Christoffel tensor is in general
\[ R^\gamma_{\gamma\alpha\beta} = \frac{\partial}{\partial x^\alpha} \Gamma^\gamma_{\gamma\beta} - \frac{\partial}{\partial x^\beta} \Gamma^\gamma_{\gamma\alpha} + \Gamma^\gamma_{\alpha\kappa} \Gamma^\kappa_{\gamma\beta} - \Gamma^\gamma_{\beta\kappa} \Gamma^\kappa_{\gamma\alpha} \]

Q: Compute one non-zero component (no sum)
\[ R^\phi_{\theta\phi\phi} = \ldots = \sin^2 \theta \]

Q: Compute (no sum)
\[ R^\theta_{\theta\phi\phi} \]

Q: Compute the Ricci tensor. Answer:
\[ R^\theta_{\theta} = R^\phi_{\phi} = a^{-2} \]
\[ R^\theta_{\phi} = R^\phi_{\theta} = 0 \]

Q: Compute the curvature scalar $R$
**Bianchi identity**

Bianchi’s identity: The curvature induced change of a vector carried over the 6 faces of a cube is zero.

Proof: Each side is traversed twice (or 4 times) in opposite directions.

In equation form:

The change on the y-z at face at $x$ is

$$d A_\sigma(x) = -A_\sigma R^{\tau}_{\gamma\gamma\zeta}(x) \, dy \, dz$$

The change on the y-z face at $x + dx$ is

$$d A_\sigma(x + dx) = -A_\sigma R^{\tau}_{\gamma\gamma\zeta}(x + dx) \, dy \, dz$$

The change over both faces is

$$d A_\sigma(x + dx) - d A_\sigma(x) = -A_\sigma \nabla_\tau R^{\tau}_{\gamma\gamma\zeta} \, dx \, dy \, dz$$

Q: Is $\nabla_\tau R^{\tau}_{\gamma\gamma\zeta}$ the same as $\frac{\partial}{\partial x} R^{\tau}_{\gamma\gamma\zeta}$?

Traverse the face at $x + dx$ in the outward-pointing sense and the face at $x$ in the outward-pointing sense.

The change over all 6 faces is

$$A_\sigma \, dx \, dy \, dz \left( \nabla_\tau R^{\tau}_{\gamma\gamma\zeta} + \nabla_\rho R^{\tau}_{\gamma\gamma\zeta} + \nabla_\chi R^{\tau}_{\gamma\gamma\zeta} \right)$$

and since each side in traversed in opposite directions, it is zero.

We chose $x$, $y$, and $z$, but we could have also chosen $t$ for one of the directions. Therefore, we have proved the Bianchi identity,

$$\nabla_\sigma R^{\tau\rho\nu\zeta} + \nabla_\rho R^{\tau\rho\nu\zeta} + \nabla_\chi R^{\tau\rho\nu\zeta} = 0$$

A contracted form of the Bianchi identity is:

$$\nabla_\rho (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0$$
In[103]:= fig[] := Module[{}, Show[Graphics3D[{Opacity[.2], Cuboid[]}, Arrow[{{{.1, .1, .1}, {.9, .1, .1}, {.9, .9, .1}, {.1, .9, 1.1}, {.1, .2, 1.1}}], Arrow[Reverse@{{.1, .1, .1}, {.9, .1, .1}, {.9, .9, .1}, {.1, .9, .1}, {.1, .2, .1}}], Arrow[{{{.1, .1, .1}, {.1, .1, 1.1}}}, Arrow[{{{.1, .2, 1.1}, {.1, .2, .2}}}, ImageSize -> 200]]

In[104]:= fig[]