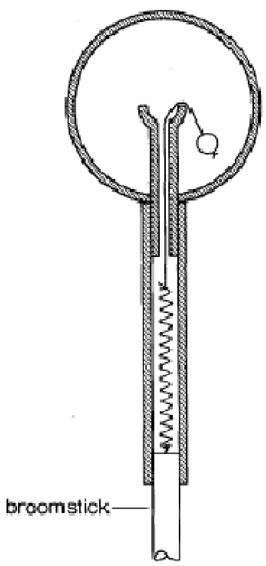
Schwarzschild metric—30 Mar 2010

- Outline
    - Einstein's toy / Foundation of Einstein's derivation of his field equation
    - Derivation of the Schwarzschild metric
- New homework will be ready tonight on Angel. Due Thurs., 8 April.

◄ | ►

Einstein's toy

"While I was living in Princeton, my wife and I would from time to time take a small puzzle involving physics to our neighbour Professor Einstein often as a birthday present. The last of these, presented on his seventy-sixth birthday, was, I believe, original. It was derived from an old-fashioned toy for small children: a ball on a string is tied to a cup in which the child has to catch the ball. But our modification was for Einstein a problem which he enjoyed, and solved at once.
A metal ball attached to a smooth thread is enclosed in a transparent globe. There is a central, transparent, cup in which the ball could rest; but initially the ball hangs by the thread outside the cup (as shown in the diagram). The thread runs from the ball up to the rim of the cup and down through a central pipe. Below the globe the thread is tied to a long, weak, spiral spring protected by a transparent tube which ends in a long pole broom-handle."
—Eric Rogers, in *Einstein, A Centenary Volume*, A. P. French, ed., Harvard, 1979, p. 131.



Q: How do you get the ball in the cup?
Q: How is this related to $G_{\mu\nu} = -8\,\pi\,G\,T_{\mu\nu}$?

The problem of the Schwarzschild metric

Schwarzschild's formulation of the problem: What is the metric outside a spherically symmetric, static star?

1. The metric does not change with time.
2. The metric is spherically symmetric.
3. The metric must be the same as Newton's gravity far from the star.

Recall (4 Mar) we found in the Newtonian limit that

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

Here $\phi = -\mathrm{GM}\big/\left(r\, c^2\right) = -M/r$ is the Newton's gravitational potential. Note the Newtonian limit is accurate to first order in ϕ for the time-time term but not the space-space term. That space is curved was not in Newton's laws.

4. Assume the metric is

$$\mathrm{ds}^2 = -B(r)\,\mathrm{dt}^2 + A(r)\,\mathrm{dr}^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

What is E's field equation in the space outside the star?

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R = -8\,\pi\, G\, T_{\mu\nu}$$

Contract:

$$g^{\mu\nu}\left(R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R\right) = -8\,\pi\, G\, g^{\mu\nu}\, T_{\mu\nu}$$

Q: Why is $g^{\mu\nu}\, g_{\mu\nu} = 4$?

$$R\ - \tfrac{1}{2}\, 4\, R = -8\,\pi\, G\, T^{\alpha}{}_{\alpha}.$$

Therefore

$$R_{\mu\nu} = -8\,\pi\, G\left(T_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, T^{\alpha}{}_{\alpha}\right)$$

In the space outside a star, the stress-energy tensor, which is

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

is zero.

Therefore

$$R_{\mu\nu} = 0.$$

Calculation in outline

The metric is

$$g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

The Christoffel symbols

$$\Gamma^{\sigma}{}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)$$

The Ricci tensor

$$R_{\mu\kappa} = \frac{\partial}{\partial x^{\kappa}} \Gamma^{\lambda}{}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}} \Gamma^{\lambda}{}_{\mu\kappa} + \Gamma^{\eta}{}_{\mu\lambda} \Gamma^{\lambda}{}_{\kappa\eta} - \Gamma^{\eta}{}_{\mu\kappa} \Gamma^{\lambda}{}_{\lambda\eta}$$

The condition $R_{\mu\nu} = 0$ imposes consitions on $A(r)$ and $B(r)$.

◄ | ►

Calculation in detail

The metric is

$$g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

The Christoffel symbols

$$\Gamma^{\sigma}{}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)$$

The Ricci tensor

$$R_{\mu\kappa} = \frac{\partial}{\partial x^{\kappa}} \Gamma^{\lambda}{}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}} \Gamma^{\lambda}{}_{\mu\kappa} + \Gamma^{\eta}{}_{\mu\lambda} \Gamma^{\lambda}{}_{\kappa\eta} - \Gamma^{\eta}{}_{\mu\kappa} \Gamma^{\lambda}{}_{\lambda\eta}$$

Christoffel symbols

On homework, we calculated the Christoffel symbols for a 2-d spherical metric. More of the the same work.

Q: Compute $\Gamma^{r}{}_{r\,r} = \frac{A'}{2A}$

$$\Gamma^{r}{}_{r\,r} = \frac{1}{2} g^{\nu\,r}\left(g_{r\,\nu,\,r} + g_{r\,\nu,\,r} - g_{\nu\,r,\,r}\right)$$

Choose a term and compute it.

$$\Gamma^{r}{}_{r\,r} = \frac{1}{2} A^{-1} A' + \frac{1}{2} A^{-1} A' - \frac{1}{2} A^{-1} A'$$

$$\Gamma^{r}{}_{\theta\theta} = -\frac{r}{A}$$

See Hartle, pp. 546–547 for the other Christoffel symbols of a spherically symmetric metric. In Hartle,

$$A(r) = e^{\nu(r,t)}$$
$$B(r) = e^{\lambda(r,t)}$$

Of course, here $\frac{\partial}{\partial t} \nu(r,\, t) = 0$ and $\frac{\partial}{\partial t} \lambda(r,\, t) = 0$.

Ricci tensor (Hartle p. 547)

$$R_{r\,r} = \frac{1}{2} \frac{B''}{B} - \frac{1}{4} \frac{B'}{B} \left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r} \frac{A'}{A}$$
$$R_{t\,t} = \frac{-1}{2} \frac{B''}{A} + \frac{1}{4} \frac{B'}{A} \left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r} \frac{B'}{A}$$
$$R_{\theta\theta} = -1 + \frac{r}{2A} \left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A}$$
$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta}$$

The non-diagonal terms are zero.

Get rid of B'':

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = \frac{-1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$$

Then

$$d \log A + d \log B = d \log (A\,B) = 0$$

Therefore

$$A\,B = \text{constant.}$$

At $r \to \infty$, the Newtonian limit yields

$$A\,B \to 1 - 2\,M/r \to 1$$

Therefore

$$A\,B = 1.$$

Consider

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A}$$

$$-1 + \frac{r}{2A}\left(2\,\frac{B'}{B}\right) + \frac{1}{A}$$

Since $A\,B = 1$,

$$-1 + r\,B' + B$$

$$= \quad -1 + \frac{d}{dr}(r\,B) = 0$$

Integrate to get

$$r\,B = r + \text{const}$$

Therefore

$$B(r) = 1 + \frac{\text{const}}{r}$$

At ∞,

$$B(r) \to 1 + 2\,\phi = 1 - 2\,\frac{M}{r}$$

Finally,

$$B(r) = 1 - 2\,\frac{M}{r}$$

$$A(r) = \left(1 - 2\,\frac{M}{r}\right)^{-1}.$$

Schwarzschild's assumption of the form of the metric

Schwarzschild's assumption of the form of the metric

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

is convenient but not fundamental.

Assume the metric does not depend on time and it depends on space $\vec{x}$ and $\overrightarrow{dx}$ only through the spatial scalars $\vec{x}\cdot\vec{x}$, $\vec{x}\cdot\overrightarrow{dx}$, and $\overrightarrow{dx}\cdot\overrightarrow{dx}$. The distance^2 depends on all possible quadratic combinations of $\left(dt,\ \overrightarrow{dx}\right)$

$$ds^2 = -F(r)\,dt^2 + 2\,E(r)\,dt\ \vec{x}\cdot\overrightarrow{dx} + D(r)\left(\vec{x}\cdot\overrightarrow{dx}\right)^2 + C(r)\,dx^2$$

Use spherical coordinates.

$$\left(x^1,\ x^2,\ x^3\right) = r(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta).$$

Then

$$ds^2 = -F(r)\,dt^2 + 2\,r\,E(r)\,dt\,dr + D(r)\,r^2\,dr^2 + C(r)\,r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

Since the metric is static, we are free to replace t by $t' + f(r)$. The t-t and t-r terms are

$$-F(r)\,(dt' + f'(r)\,dr)^2 + 2\,r\,E(r)\,dr\,(dt' + f'(r)\,dr)$$

If we choose

$$-2\,F(r)\,f'(r) = 2\,r\,E(r),$$

then the dt dr term is zero, and

$$ds^2 = -F(r)\,dt^2 + G(r)\,r^2\,dr^2 + C(r)\,r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

where $G(r) = r^2\left[D(r) + E^2(r)\big/F(r)\right]$

We may change the r coordinate. $r'^2 = r^2\,C(r)$. Then

$$ds^2 = -B(r')\,dt^2 + A(r')\,dr'^2 + r'^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)$$

where

$$B(r') = F(r)$$

$$A(r') = [1 + G(r)/C(r)][1 + r/(2\,C(r))\,C'(r)]^{-2}.$$