Robertson-Walker metric—1 Apr 2010

A 3-d space that is homogeneous and isotropic has a special choice of time. A choice of coordinates is 

$$(t, r, \theta, \phi)$$

and the metric is

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - (r/r_0)^2} + r^2(\sin^2 \theta d\phi^2) \right]$$

$(r, \theta, \phi)$ is called the comoving coordinate. A galaxy stays at the same position; time changes.

$r_0^2$ can have any value, positive or negative

$a(t)$ is called the expansion parameter.

Friedman's equation is

$$\left( \frac{1}{a} \frac{da}{dt} \right)^2 = \frac{8\pi}{3} G \rho = -\frac{1}{a^2}$$

We derived Friedmann's equation except for the constant $-\frac{1}{r_0^2}$ on the RHS.

- **Outline**
  - Derivation of the Robertson-Walker metric
  - Derivation of Friedman's equation from the Robertson-Walker metric
From of the metric

Assumptions:
1) There is a time coordinate that is proper time.
2) At a given time, the space within a small bubble is isotropic.
3) At a given time, the space is homogeneous.

The metric is
\[ ds^2 = -dt^2 + A(t, \text{vector } r) \left( dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \]

Q: What assumptions have gone into writing a metric of this form?

Since the space is homogeneous, \( A(t, r_1)/A(t, r_2) \) can depend on time and also on the distance between the points. It cannot depend on the location. Therefore
\[ ds^2 = -dt^2 + a(t)^2 B(r) \left( dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \]

Q: Have I eliminated the possibility of a curved space by grouping the \( dr, d\theta, \) and \( d\phi \) terms together as \( (dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \), which is the case for flat, spherical coordinates?

By mapping the 3-d onto the surface of a 4-d symmetric space, we can show that the possible metrics are
\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - \left( \frac{r}{r_0} \right)^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \]
\[ ds^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 + \left( \frac{r}{r_0} \right)^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \]

and
\[ ds^2 = -dt^2 + a(t)^2 \left( dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \]

Q: Interpret the spatial parts of this metric.
Derivation of Friedman’s equation

The metric is
\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - r_0/r} + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right] \]

Since the coordinate time is the same as proper time, in average the galaxies are at rest.

The stress-energy tensor
\[ T^{\mu \nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \]

Q: Galaxies occur in clusters. If the motion of the galaxies is faster, does that change the stress-energy tensor?

Einstein’s equation is
\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = -8 \pi G T_{\mu \nu} \]

Plan: Calculate the curvature tensors and find conditions on \( a(t) \) and \( r_0 \) that satisfy E’s equations.

Rewrite
\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - r_0/r} + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right] \]
as
\[ ds^2 = -dt^2 + a(t)^2 (\tilde{g}_{11} \, dx_1^2 + \tilde{g}_{22} \, dx_2^2 + \tilde{g}_{33} \, dx_3^2) \]

The derivatives of \( \tilde{g} \) are not zero. At small \( r \), \( \tilde{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

1. Easy one: We found (on 3/31) that the curvature scalar
\[ R = 4 \pi G T_{\alpha \beta}. \]

Here
\[ T_{\alpha \beta} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & a^2 P & 0 & 0 \\ 0 & 0 & a^2 P & 0 \\ 0 & 0 & 0 & a^2 P \end{pmatrix} \]
\[ R = 4 \pi G (\rho - 3 a^2 P). \]
Derivation of Friedman’s equation, part 2

2. Compute Christoffel symbols

a) Time part

\[ \Gamma^0_{\alpha\beta} = \frac{1}{2} \left( g^{00} \left( g_{0,\alpha,\beta} + g_{0,\beta,\alpha} - g_{\alpha,\beta,0} \right) \right) \]

\[ \alpha = 0: \]

\[ \Gamma^0_{\alpha\beta} = \frac{1}{2} \left( g^{00} \left( g_{0,\alpha,\beta} + g_{0,\beta,\alpha} - g_{\alpha,\beta,0} \right) \right) = \frac{1}{2} \left( g^{00} \right) = 0, \]

because \( g^{00} = -1 \).

\[ \alpha = i \neq 0 \text{ and } \beta = j \neq 0: \]

\[ \Gamma^0_{ij} = \frac{1}{2} \left( g^{00} \right) \left( g_{i,j,0} + g_{0,j,i} - g_{i,j,0} \right) \]

The first two terms are zero because ???.

\[ \Gamma^0_{ij} = -\frac{1}{2} \left( -g_{ij,0} \right) = \alpha(t) \frac{da}{dt} \tilde{g}_{ij}. \]

b) Space part

\[ \Gamma^i_{\alpha\beta} = \frac{1}{2} g^{ij} \left( g_{\alpha,\beta} + g_{\beta,\alpha} - g_{\alpha,\beta,0} \right) \]

\[ \alpha = 0: \]

\[ \Gamma^i_{0\beta} = \frac{1}{2} g^{ij} \left( g_{0,\beta} + g_{\beta,0} - g_{0,\beta,0} \right) \]

The first term is zero. Second term is zero unless \( \beta = i \), in which case it is \( \frac{1}{2} g^{ii} a \frac{da}{dt} \tilde{g}_{ii} = \frac{\dot{a}}{a} \). The third term is zero, since \( g_{00} = -1 \).

\[ \Gamma^i_{0\beta} = \frac{\dot{a}}{a} \delta^i_\beta. \]

\[ \alpha = i \neq 0 \text{ and } \beta = j \neq 0: \]

\[ \Gamma^i_{jk} = \frac{1}{2} g^{ij} \left( g_{ij,k} + g_{ik,j} - g_{jk,i} \right) \]

involves space coordinates only.