Derivation of Friedman's equation—6 Apr 2010

The metric is

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{d\,r^2}{1-(r/r_0)^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Since the coordinate time is the same as proper time, in average the galaxies are at rest.

Q: Galaxies occur in clusters. If the motion of the galaxies is faster, does that change the stress-energy tensor?

Einstein's equation is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = -8\,\pi\, G\, T_{\mu\nu}$$

Plan: Calculate the curvature tensors and find conditions on $a(t)$ and r_0 that satisfy E's equations.

Rewrite

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{d\,r^2}{1-(r/r_0)^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

as

$$ds^2 = -dt^2 + a(t)^2 \left(\tilde{g}_{rr}\, dr^2 + \tilde{g}_{\theta\theta}\, d\theta^2 + \tilde{g}_{\phi\phi}\, d\phi^2\right)$$

where $\tilde{g} = \begin{pmatrix} \left[1-(r/r_0)^2\right]^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

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Compute the Christoffel symbols

a) Time term

$$\Gamma^0{}_{\alpha\beta} = \frac{1}{2} g^{00}\left(g_{0\,\alpha,\,\beta} + g_{0\,\beta,\,\alpha} - g_{\alpha\beta,\,0}\right)$$

$\alpha = 0$:

$$\Gamma^0{}_{0\,\beta} = \frac{1}{2} g^{00}\left(g_{00,\,\beta} + g_{0\,\beta,\,0} - g_{0\,\beta,\,0}\right) = \frac{1}{2} g^{00}\, g_{00,\,\beta} = 0,$$

because $g_{00} = -1$.

$\alpha = i \neq 0$ and $\beta = j \neq 0$:

$$\Gamma^0{}_{ij} = \frac{1}{2} g^{00}\left(g_{0\,i,\,j} + g_{0\,j,\,i} - g_{ij,\,0}\right)$$

The first two terms are zero because ???.

$$\Gamma^0{}_{ij} = -\frac{1}{2}\left(-g_{ij,\,0}\right) = a(t)\,\frac{da}{dt}\,\tilde{g}_{\mathrm{ij}}.$$

$$\boxed{\Gamma^0{}_{ij} = \dot{a}\;a\,\tilde{g}_{ij}}$$

b) Space-time term: $\alpha = 0$ in

$$\Gamma^i{}_{\alpha\beta} = \frac{1}{2} g^{ii}\left(g_{\mathrm{i}\alpha,\,\beta} + g_{\mathrm{i}\beta,\,\alpha} - g_{\alpha\beta,\,i}\right)$$

$$\Gamma^i{}_{0\,\beta} = \frac{1}{2} g^{ii}\left(g_{\mathrm{i}0,\,\beta} + g_{\mathrm{i}\beta,\,0} - g_{0\,\beta,\,i}\right)$$

The first term is zero. Second term is zero unless $\beta = i$, in which case it is $\frac{1}{2} g^{\mathrm{ii}}\, 2\, a\, \dot{a}\, \tilde{g}_{\mathrm{ii}} = \frac{\dot{a}}{a}$. The third term is zero, since $g_{00} = -1$.

$$\boxed{\Gamma^i{}_{0\,\beta} = \Gamma^i{}_{\beta 0} = \frac{\dot{a}}{a}\,\delta^i_\beta}$$

c) Space-space term: $\alpha = i \neq 0$ and $\beta = j \neq 0$ in

$$\Gamma^i{}_{\mathrm{jk}} = \frac{1}{2} g^{\mathrm{ii}}\left(g_{\mathrm{ij},\,k} + g_{\mathrm{ik},\,j} - g_{\mathrm{jk},\,i}\right)$$

involves space coordinates only. Straightforward. Hartle p. 547

$$\boxed{\Gamma^r{}_{rr} = \frac{r}{r_0^2}\,\frac{1}{1-\left(r/r_0\right)^2}} \quad \boxed{\Gamma^r{}_{\theta\theta} = -r\left[1 - (r/r_0)^2\right]} \quad \boxed{\Gamma^r{}_{\phi\phi} = -r\sin^2\theta\left[1 - (r/r_0)^2\right]} \quad \boxed{\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = 1/r}$$

$$\boxed{\Gamma^\theta{}_{\phi\phi} = -\cos\theta\sin\theta} \quad \boxed{\Gamma^\phi{}_{\theta\phi} = \cot\theta}$$

The other terms are zero.

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Form of the Ricci tensor and the source of curvature

What is the form of the space part of the Ricci tensor in the frame in which the matter is at rest? Recall that the universe is isotropic and homogeneous.

Q: Take a guess.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8 \pi G T_{\mu\nu}$$

Take the trace

$$R^{\mu}{}_{\mu} - \frac{1}{2} 4 R = -8 \pi G T^{\mu}{}_{\mu}$$

to find that the curvature scalar is related to the trace of the stress-energy tensor:

$$R = 8 \pi G T$$

$$\boxed{R_{\mu\nu} = -8 \pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)}$$

The stress-energy tensor

$$T^{\mu\nu} = (\rho + P) u^{\mu} u^{\nu} + g^{\mu\nu} P$$

For the frame in which the galaxies are at rest, $u^{\mu} = (1, 0, 0, 0)$.

$$T^{\mu\nu} = \text{diagonal}\left\{\rho, a^{-2}\left[1-(r/r_0)^2\right] P, a^{-2} r^{-2} P, (a r \sin\theta)^{-2} P\right\}.$$

$$T^{\mu}{}_{\nu} = T^{\mu\alpha} g_{\alpha\nu} = \text{diagonal}(-\rho, P, P, P)$$

$$T = T^{\mu}{}_{\mu} = -\rho + 3 P$$

$$T_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} T^{\mu\nu} = \text{diagonal}\left\{\rho, a^2\left[1-(r/r_0)^2\right]^{-1} P, a^2 r^2 P, (a r \sin\theta)^2 P\right\}$$

The source of the curvature

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$$

$$= \frac{1}{2} \text{diagonal}\left\{\rho + 3 P, a^2\left[1-(r/r_0)^2\right]^{-1} (\rho - P), a^2 r^2(\rho - P), (a r \sin\theta)^2 (\rho - P)\right\}$$

Then

$$\boxed{S_{tt} = \frac{1}{2} (\rho + 3 P)}$$

$$\boxed{S_{ij} = \frac{1}{2} (\rho - P) a^2 \tilde{g}_{ij}}$$

Since $R_{\mu\nu} = -8 \pi G S_{\mu\nu}$, the space-space part of the Ricci tensor is

$$\boxed{R_{ij} = f(t)\ \tilde{g}_{ij}}$$

even though it involves derivatives of the metric.

Computation of Ricci tensor

Plan:
a. Compute R_{tt}.
b. Compute $R_{rr}$to find $f(t)$.

a) $R_{00} = -\Gamma^{\alpha}{}_{00,\alpha} + \Gamma^{\alpha}{}_{0\,\alpha,0} + \Gamma^{\alpha}{}_{\sigma 0}\,\Gamma^{\sigma}{}_{0\,\alpha} - \Gamma^{\alpha}{}_{\sigma\alpha}\,\Gamma^{\sigma}{}_{00}$

1st and 4th terms are zero because $\Gamma^{\alpha}{}_{00} = 0$.

2nd term: $\Sigma_i\,\Gamma^{i}{}_{0\,i,0} = \Sigma_i\,\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = 3\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right)$

3rd term: If $\alpha = 0$, it is 0. $\Sigma_{i,j}\,\Gamma^{i}{}_{j0}\,\Gamma^{j}{}_{0\,i} = \Sigma_i\left(\Gamma^{i}{}_{i0}\right)^2 = 3\,\frac{\dot{a}}{a}$

Therefore

$$\boxed{R_{00} = 3\,\frac{\ddot{a}}{a}}$$

b) $R_{rr} = -\Gamma^{\alpha}{}_{rr,\alpha} + \Gamma^{\alpha}{}_{r\,\alpha,r} + \Gamma^{\alpha}{}_{\sigma\,r}\,\Gamma^{\sigma}{}_{r\,\alpha} - \Gamma^{\alpha}{}_{\sigma\alpha}\,\Gamma^{\sigma}{}_{rr}$

$= \; -\Gamma^{t}{}_{rr,t} + \Gamma^{t}{}_{r\,t,r} + \Gamma^{t}{}_{\sigma\,r}\,\Gamma^{\sigma}{}_{r\,t} - \Gamma^{t}{}_{\sigma\,\alpha}\,\Gamma^{\sigma}{}_{r\,r}$

$= \; -\left(+\Gamma^{r}{}_{rr,r} + \Gamma^{\theta}{}_{rr,\theta} + \Gamma^{\phi}{}_{rr,\phi}\right)$

$+\left(\Gamma^{r}{}_{r\,\alpha,r} + \Gamma^{\theta}{}_{r\,\theta,r} + \Gamma^{\phi}{}_{r\,\phi,r}\right) + \Gamma^{\alpha}{}_{\sigma\,r}\,\Gamma^{\sigma}{}_{r\,\alpha} - \Gamma^{\alpha}{}_{\sigma\alpha}\,\Gamma^{\sigma}{}_{rr}$

Collect the terms involving time:

$$-\Gamma^{t}{}_{rr,t} + \Gamma^{t}{}_{r\,t,r} + \Gamma^{t}{}_{\sigma\,r}\,\Gamma^{\sigma}{}_{r\,t} - \Gamma^{t}{}_{\sigma\,t}\,\Gamma^{\sigma}{}_{r\,r} + \Gamma^{r}{}_{t\,r}\,\Gamma^{t}{}_{r\,r} - \left(\Gamma^{r}{}_{t\,r} + \Gamma^{\theta}{}_{t\,\theta} + \Gamma^{\phi}{}_{t\,\phi}\right)\Gamma^{t}{}_{rr}$$
$$= \; -\Gamma^{t}{}_{rr,t} + 0 + \Gamma^{t}{}_{r\,r}\,\Gamma^{r}{}_{r\,t} - 0 + \Gamma^{r}{}_{t\,r}\,\Gamma^{t}{}_{r\,r} - (3\,\Gamma^{r}{}_{t\,r})\,\Gamma^{t}{}_{rr}$$
$$= \; -\Gamma^{t}{}_{rr,t} - \Gamma^{t}{}_{r\,r}\,\Gamma^{r}{}_{r\,t}$$
$$= \; -\frac{\partial}{\partial t}\,\dot{a}\,a\,\tilde{g}_{r\,r} - \left(\dot{a}\,a\,\tilde{g}_{r\,r}\right)(\dot{a}\,/a)$$
$$= \; -\left(\ddot{a}\;a + 2\,\dot{a}^2\right)\tilde{g}_{r\,r}$$

Then

$$\boxed{R_{rr} = -\left(\ddot{a}\;a + 2\,\dot{a}^2\right)\tilde{g}_{r\,r} + \tilde{R}_{r\,r}}$$

where

$$\tilde{R}_{r\,r} = -\Gamma^{i}{}_{rr,i} + \Gamma^{i}{}_{r\,i,r} + \Gamma^{i}{}_{j\,r}\,\Gamma^{j}{}_{r\,i} - \Gamma^{i}{}_{j\,i}\,\Gamma^{j}{}_{rr}$$

c) Compute $\tilde{R}_{r\,r}$.

The terms with derivatives:

$$-\Gamma^{i}{}_{rr,i} + \Gamma^{i}{}_{r\,i,r} = -\Gamma^{r}{}_{rr,r} + \left(\Gamma^{r}{}_{rr,r} + \Gamma^{\theta}{}_{r\theta,r} + \Gamma^{\phi}{}_{r\phi,r}\right) = 2\,\frac{d}{dr}\,r^{-1} = -\frac{2}{r^2}$$

The other terms:

$$\Gamma^i{}_{jr}\,\Gamma^j{}_{ri} - \Gamma^i{}_{ji}\,\Gamma^j{}_{rr} = \Gamma^r{}_{rr}\,\Gamma^r{}_{rr} + \Gamma^\theta{}_{r\theta}\,\Gamma^\theta{}_{r\theta} + \Gamma^\phi{}_{r\phi}\,\Gamma^\phi{}_{r\phi} - \left(\Gamma^r{}_{rr} + \Gamma^\theta{}_{\theta r} + \Gamma^\phi{}_{\phi r}\right)\Gamma^r{}_{rr}$$

$$= \;2\,\frac{1}{r^2} - 2\,\frac{1}{r}\,\frac{r}{r_0^2}\,\frac{1}{1-(r/r_0)^2}$$

Finally,

$$\tilde{R}_{rr} = -\frac{2}{r^2} + \frac{2}{r^2} - \frac{2}{r_0^2}\,\frac{1}{1-(r/r_0)^2}$$

$$= \;-2\,r_0^{-2}\,\tilde{g}_{rr}$$

The Christoffel symbol involves first derivatives of the metric tensor. Ricci and Riemann curvature tensors involve second derivatives, and therefore their units are distance^{-2} or time^{-2}.
The space-space part of the Ricci tensor is $-2\,(\text{radius curvature})^{-2}$ metric. In n dimensions, the factor 2 is different, but the form is the same.

Since this is true for all of the diagonal element, not only for the r-r term,

$$\boxed{R_{ii} = -\left(\ddot{a}\;a + 2\,\dot{a}^2 + 2\,r_0^{-2}\right)\tilde{g}_{ii}}$$

Friedman's equation

Put Ricci tensor into Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) = -8\pi G S_{\mu\nu}$$

The time-time term:

$$3\frac{\ddot{a}}{a} = -8\pi G \frac{1}{2}(\rho + 3P)$$

$$\ddot{a} = -\frac{4\pi}{3} G(\rho + 3P) a$$

Q: Interpret the result for the time-time part of Einstein's field equation appied to the R-W metric.

The i-i term:

$$-\left(\ddot{a}\, a + 2\dot{a}^2 + \frac{2}{r_0^2}\right)\tilde{g}_{ii} = -8\pi G \frac{1}{2}(\rho - P) a^2 \tilde{g}_{ii}$$

Eliminate $\ddot{a}$ to get

$$\dot{a}^2 - \frac{8\pi}{3} G \rho a^2 + \frac{1}{r_0^2} = 0$$

Q: Interpret the result.

Q: What is the big surprise?

Q: In outline, how did we derive this result?

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Gravitational waves—6 April 2010

- Outline
  - Introduction: metric, polarization, how to detect (§16)
  - Wave equation (§21.5)
  - Source of gravitational waves (§23)

Metric for gravitational waves

For a particular gravitational wave, the metric is

$$ds^2 = -dt^2 + [1 + f(t - z)]\, dx^2 + [1 - f(t - z)]\, dy^2 + dz^2$$

where $f(t - z) \ll 1$.

Q: In what way is there a wave?

Q: In which direction is the wave moving?

Q: What is the speed of the wave?

Q: How does the wave affect distances?