Equation for weak gravity waves—13 April 2010

- Outline
  - Introduction (§16)
    How to detect gravity waves
    Order-of-magnitude strains
    Polarization
  - Wave equation (§21.5) (Today)
  - Source of gravitational waves (§23)

For gravitational waves, the perturbation of the metric is small. Let the metric be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is small.

Plan:

1) Compute the Christoffel symbols and then the Ricci tensor. Keep first-order terms in h. Then use Einstein's equation

$$R_{\mu\nu} = -8\pi G S_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right).$$

Will get the wave equation.

2) Solve the wave equation for plane waves.

E's equation for weak fields

The Christoffel symbols

$$\Gamma^{\sigma}{}_{\lambda\mu} = \frac{1}{2}\, g^{\nu\sigma}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)$$

The terms $g_{\mu\nu,\lambda}$are first order in h. Therefore we neglect h in the term $g^{\nu\sigma}$.

$$\Gamma^{\sigma}{}_{\lambda\mu} = \frac{1}{2}\, \eta^{\nu\sigma}\left(h_{\mu\nu,\lambda} + h_{\lambda\nu,\mu} - h_{\mu\lambda,\nu}\right)$$

The Ricci tensor

$$R_{\mu\kappa} = \frac{\partial}{\partial x^{\kappa}}\, \Gamma^{\lambda}{}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}}\, \Gamma^{\lambda}{}_{\mu\kappa} + \Gamma^{\eta}{}_{\mu\lambda}\, \Gamma^{\lambda}{}_{\kappa\eta} - \Gamma^{\eta}{}_{\mu\kappa}\, \Gamma^{\lambda}{}_{\lambda\eta}$$

Neglect the 3rd and 4th terms are 2^{nd} order in h.

$$R_{\mu\kappa} = \frac{\partial}{\partial x^{\kappa}}\, \Gamma^{\lambda}{}_{\mu\lambda} - \frac{\partial}{\partial x^{\lambda}}\, \Gamma^{\lambda}{}_{\mu\kappa}$$

Now do the work.

Term $\frac{\partial}{\partial x^{\kappa}}\, \Gamma^{\lambda}{}_{\mu\lambda}$:

$$\Gamma^{\lambda}{}_{\lambda\mu} = \frac{1}{2}\, \eta^{\nu\lambda}\left(h_{\mu\nu,\lambda} + h_{\lambda\nu,\mu} - h_{\mu\lambda,\nu}\right)$$

Since I can swap ν and λ on RHS, 1st and 3rd terms cancel.

$$\frac{\partial}{\partial x^{\kappa}}\, \Gamma^{\lambda}{}_{\mu\lambda} = \frac{1}{2}\, \eta^{\nu\lambda}\, \frac{\partial^2}{\partial x^{\mu}\, \partial x^{\kappa}}\, h_{\lambda\nu} = \frac{1}{2}\, \frac{\partial^2}{\partial x^{\mu}\, \partial x^{\kappa}}\, h^{\lambda}_{\lambda}$$

The other term

$$-\frac{\partial}{\partial x^{\lambda}}\, \Gamma^{\lambda}{}_{\mu\kappa} = -\frac{1}{2}\, \eta^{\nu\lambda}\left(h_{\mu\nu,\kappa,\lambda} + h_{\kappa\nu,\mu,\lambda} - h_{\mu\kappa,\nu,\lambda}\right)$$

$$= \ -\frac{1}{2}\left(h^{\lambda}_{\mu,\kappa,\lambda} + h^{\lambda}_{\kappa,\mu,\lambda} - h^{,\lambda}_{\mu\kappa,,\lambda}\right)$$

The whole works:

$$R_{\mu\kappa} = -\frac{1}{2}\left(\frac{\partial}{\partial x^{\kappa}}\, A_{\mu} + \frac{\partial}{\partial x^{\mu}}\, A_{\kappa} - h^{,\lambda}_{\mu\kappa,\lambda}\right),$$

where

$$A_{\mu} = \frac{\partial}{\partial x^{\lambda}}\, h^{\lambda}_{\mu} - \frac{1}{2}\, \frac{\partial}{\partial \mu}\, h^{\lambda}_{\lambda}.$$

In more customary notation,

$$h^{,\lambda}_{\mu\kappa,\lambda} = \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) h_{\mu\kappa}$$

is the d'Alembertian $\Box\, h_{\mu\kappa}$.

Finally,

$$\boxed{R_{\mu\kappa} = -\frac{1}{2}\left(\frac{\partial}{\partial x^{\kappa}}\, A_{\mu} + \frac{\partial}{\partial x^{\mu}}\, A_{\kappa} - \Box\, h_{\mu\kappa}\right) = -8\,\pi\, G\, S_{\mu\nu}}$$

Q: Is this a customary wave equation? What are the terms?

Unphysical freedom

There is freedom in h. Change the coordinate system. Replace x^α by

$$x'^\alpha = x^\alpha + \epsilon^\alpha(x),$$

where $\epsilon^\alpha(x)$ is small just like $h_{\alpha\beta}$. Then

$$\eta^{\alpha\beta} + h'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} (\eta^{\mu\nu} + h^{\mu\nu})$$

$$= \left(\delta^\alpha_\mu + \frac{\partial \epsilon^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu + \frac{\partial \epsilon^\beta}{\partial x^\nu}\right)(\eta^{\mu\nu} + h^{\mu\nu})$$

$$= \eta^{\alpha\beta} + h^{\alpha\beta} + \frac{\partial \epsilon^\alpha}{\partial x^\mu} \eta^{\mu\beta} + \frac{\partial \epsilon^\beta}{\partial x^\nu} \eta^{\alpha\nu}$$

I ignored the 2nd-order terms to write the last line. The 1st order term is

$$h'^{\alpha\beta} = h^{\alpha\beta} + \frac{\partial \epsilon^\alpha}{\partial x^\mu} \eta^{\mu\beta} + \frac{\partial \epsilon^\beta}{\partial x^\nu} \eta^{\alpha\nu}$$

Upon a change in the coordinate system, the strain changes to

$$\boxed{h'_{\alpha\beta} = h_{\alpha\beta} + \frac{\partial \epsilon_\alpha}{\partial x^\beta} + \frac{\partial \epsilon_\beta}{\partial x^\alpha}}$$

Choose a "gauge" so that

$$\boxed{A_\mu = \frac{\partial}{\partial x^\lambda} h^\lambda_\mu - \frac{1}{2} \frac{\partial}{\partial x^\mu} h^\lambda_\lambda = 0}.$$

If h does not satisfy this condition, then choose a transformation $\epsilon(x)$ so that it does.
Then we have the wave equation

$$\boxed{\Box h_{\mu\kappa} \equiv \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) h_{\mu\kappa} = -16\,\pi\,G\,S_{\mu\kappa}}.$$

Solution of the wave equation

Guess a plane-wave solution

$$h_{\alpha\beta}(x^\gamma) = a_{\alpha\beta}\, e^{i\, k_\gamma\, x^\gamma}.$$

Q: How do you take derivatives of h? Compute $\frac{\partial^2}{\partial t^2} h_{\alpha\beta}$.

■

For this to satisfy the wave equation,

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)a_{\alpha\beta}\,e^{\iota\,k_\gamma\,x^\gamma}=a_{\alpha\beta}\,e^{\iota\,k_\gamma\,x^\gamma}\left(-k_0^2+k_x^2+k_y^2+k_z^2\right)=0,$$

or for some $a_{\alpha\beta}\neq 0$,

$$k_\mu\,k^\mu=0.$$

Q: Interpret $k_\mu\,k^\mu=0$.

For the solution to satisy the gauge,

$$k_\lambda\,a^\lambda_\beta\,e^{\iota\,k_\gamma\,x^\gamma}=\frac{1}{2}\,k_\beta\,a^\lambda_\lambda\,e^{\iota\,k_\gamma\,x^\gamma}.$$

$$k_\lambda\,a^\lambda_\beta=\frac{1}{2}\,k_\beta\,a^\lambda_\lambda.$$

Problem: Consider a wave traveling in the z direction.

$$k^\mu=(k,\,0,\,0,\,k),$$

where $k>0$. Determine the possible values for $a_{\alpha\beta}$.

The gauge condition $k^\lambda\,a_{\lambda\beta}=\frac{1}{2}\,k_\beta\,a^\lambda_\lambda$.

$$k\,a_{0\,\beta}+k\,a_{3\,\beta}=\frac{1}{2}\,k_\beta(-a_{00}+a_{11}+a_{22}+a_{33})$$

gives

$$a_{00}+a_{30}=-\frac{1}{2}\,(-a_{00}+a_{11}+a_{22}+a_{33})$$

$$a_{01}+a_{31}=a_{02}+a_{32}=0$$

$$a_{03}+a_{33}=\frac{1}{2}\,(-a_{00}+a_{11}+a_{22}+a_{33})$$

So

$$a_{01}=-a_{31}$$

$$a_{02}=-a_{32}$$

$$a_{03}=-\frac{1}{2}\,(a_{33}+a_{00})$$

$$a_{22}=-a_{11}$$

Replace x^α by

$$x'^\alpha=x^\alpha+\epsilon^\alpha(x).$$

Then $\quad h'_{\alpha\beta}=h_{\alpha\beta}+\frac{\partial\epsilon_\alpha}{\partial x^\beta}+\frac{\partial\epsilon_\beta}{\partial x^\alpha}$ yields

$$a'_{11}=a_{11}+2\,k_1\,\epsilon_1=a_{11}$$

$$a'_{12}=a_{12}+k_1\,\epsilon_2+k_2\,\epsilon_1=a_{12}$$

$$a'_{13}=a_{13}+k_1\,\epsilon_3+k_3\,\epsilon_1=a_{13}+k\,\epsilon_1$$

$$a'_{23}=a_{23}+k_2\,\epsilon_3+k_3\,\epsilon_2=a_{23}+k\,\epsilon_2$$

$$a'_{33}=a_{33}+2\,k_3\,\epsilon_3=a_{33}+2\,k\,\epsilon_3$$

$$a'_{00}=a_{00}-2\,k_0\,\epsilon_0=a_{00}-2\,k\,\epsilon_0$$

because $k_1=k_2=0$.

Q: Interpret these equations. Which $a_{\alpha\beta}$ are physical?

■

I can choose ϵ_1 to eliminate a_{13}. Similarly I can eliminate a_{23}, a_{33}, and a_{00}.
Setting $a_{33} = 0$ removes the longitudinal polarization.

a_{11} and a_{12} do not change. They cannot be removed with a coordinate change.

There are two independent numbers a_{11} and a_{12}. The two independent polarizations are

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Q: Simplicio: If I did not choose to eliminate a_{33}, then there would be additional terms in the metric

$$ds^2 = -dt^2 + [1 + f(t-z)]\,dx^2 + [1 - f(t-z)]\,dy^2 + [1 + w(t-z)]\,dz^2 - w(t-z)\,dt\,dz.$$

I could make $w(t-z)$ really big and detect gravity waves easily. What is wrong?

◄ | ►

Spin of gravity waves

Consider the polarization

$$a_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rotate by angle θ about the z axis. The transformation is

$$\Lambda_1{}^1 = \cos\theta$$
$$\Lambda_1{}^2 = \sin\theta$$
$$\Lambda_2{}^1 = -\sin\theta$$
$$\Lambda_2{}^2 = \cos\theta$$
$$\Lambda_0{}^0 = \Lambda_3{}^3 = 1$$

and the other terms are 0.

$$a'_{\alpha\beta} = \Lambda_\alpha{}^\gamma \, \Lambda_\beta{}^\delta \, a_{\gamma\delta}$$

Do the 11 term

$$a'_{11} = \Lambda_1{}^\gamma \, \Lambda_1{}^\delta \, a_{\gamma\delta} = \Lambda_1{}^1 \, \Lambda_1{}^1 \, a_{11} + \Lambda_1{}^2 \, \Lambda_1{}^1 \, a_{21} + \Lambda_1{}^1 \, \Lambda_1{}^2 \, a_{12} + \Lambda_1{}^2 \, \Lambda_1{}^2 \, a_{22}$$
$$= \cos^2\theta - 2\,i\sin\theta\cos\theta - \sin^2\theta$$
$$= \cos 2\theta - i\sin 2\theta$$
$$= e^{2\,i\,\theta}$$

We can show that

$$a'_{\alpha\beta} = e^{2\,i\,\theta} \, a_{\alpha\beta}$$

For the polarization

$$b_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$b'_{\alpha\beta} = e^{-2\,i\,\theta} \, b_{\alpha\beta}$$

Q: For what rotation angle is the rotated wave the same as the original wave?

Gravity waves are spin 2.