Thermodynamic temperature of a black hole—22 Apr 2010

- Outline
  - Hints of strangeness from our study of the orbits in the Schwarzscild metric
  - Eddington-Finkelstein coordinates for the Schwarzschild metric (§12).
  - Radiation from black holes (§13.3) (Today)
  - Stellar collapse (§24)
  - Observations of black holes

■ A heat engine

A box is filled with radiation at temperature T. The box expands adiabatically and does work. It delivers heat to a bath at temperature T_c. It contracts adiabatically at gets heat from bath at temperature T. A cycle is complete.

The work $W = Q\left(1 - \frac{T_c}{T}\right)$.

The efficiency

$$\epsilon \equiv \frac{W}{Q} = 1 - \frac{T_c}{T}.$$

■ A heat engine using a black hole as the cold-temperature reservoir

1. At large r, fill a box with radiation of temperature T. The mass of the radiation is $m = E \big/ c^2$.

2. Tie the box to a string and lower the box toward the black hole to r. The string pulls on a generator. The work done is $W = 2\,\frac{M}{r}\,m$. Lowered to $r = 2\,M$, the work is

$$W = m\,c^2 = E.$$

In other words, if a mass falls to the horizon, all of its mass is converted to energy.

Q: How is this relavant for a quasars, which has a black hole?

3. Open the box to let out the radiation. Now the mass in the box is zero. Pull the box back up.

4. Fill the box with radiation of temperature T.

We have an cyclic engine, and all of the energy has been converted to work. The efficiency

$$\epsilon = 1$$

Q: Therefore the temperature of the black hole is 0.

▪ Better calculation

We made a slight error that changes the temperature of the BH. The characteristic wavelength of radiation of temperature T is given by

$$\frac{h\,c}{\lambda} = k\,T.$$

The length L of the box has to be $L > \lambda$.

The center of the box is at $r = 2\,M + \frac{1}{2}\,\frac{h\,c}{k\,T}$. The work is

$$W = 2\,M \Big/ \left(2\,M + \frac{1}{2}\,\frac{h\,c}{k\,T}\right) m\,c^2 = \left(1 - \frac{h\,c}{4\,M\,k\,T}\right) m\,c^2$$

The efficiency

$$\epsilon = \left(1 - \frac{h\,c}{4\,M\,k\,T}\right).$$

Therefore the temperature of a black hole (a more precise calculation) is

$$\boxed{T_{\mathrm{BH}} = \frac{\hbar\,c^3}{8\,\pi\,G\,M\,k}}$$

Entropy: If I add a mass δM to a black hole, the heat added is $Q = \delta\,M\,c^2$. The change in entropy is

$$\delta\,S = \frac{Q}{T} = 8\,\pi\,G\,M\,k\ \ \delta\,M / (\hbar\,c)$$

The area of a BH is

$$A = 4\,\pi\left(2\,G\,M \big/ c^2\right)^2$$

Adding mass changes the area

$$\delta\,A = 32\,\pi\,G^2\,M\ \delta\,M \big/ c^4.$$

Therefore the entropy of a black hole is proportional to its area.

$$\boxed{S = \frac{k\,c^3}{4\,\pi\,\hbar\,G}\,A}$$

Q: What effects does this idea encompass? Look at the constants.

Numerical values

Planck's constant $\hbar$ has units of length^2. This is called the Planck length

$$L_P = (\hbar G / c^3)^{1/2}$$
$$= 1.6\times 10^{-33}\ \text{cm}$$

```
<< PhysicalConstants`
```

```
Convert[PlanckConstant / (2 π) GravitationalConstant / SpeedOfLight^3, Centimeter^2]
```

```
2.61227 × 10^-66 Centimeter^2
```

```
Sqrt[% / Centimeter^2]
```

```
1.61625 × 10^-33
```

```
x = PlanckConstant / (2 π) SpeedOfLight^3 / (8 π GravitationalConstant BoltzmannConstant);
```

```
x / {Kilogram, SolarMass};
```

```
Convert[# / Kelvin, 1] & /@ %
```

```
{1.2269 × 10^23, 6.16813 × 10^-8}
```

```
x / 300
```

$$\frac{4.08968\times 10^{20}\ \text{Kelvin Kilogram}^2\ \text{Meter}}{\text{Newton Second}^2}$$

The temperature of a 1-kg black hole is $1\times 10^{23}\ K$.
The temperature of a 1 M_{sun} black hole is $6\times 10^{-8}\ K$
The mass of a room-temperature black hole is 4×10^{20} kg $\approx 10^{-4}\ M_{\text{earth}}$

The temperature of a black hole is

$$k\,T = c^3\,\hbar/(8\,\pi\,G\,M)$$

Hawking emission

Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

is invariant with a translation in t. Therefore the momentum p_t is conserved.

Q: You set up a laboratory in the BPS building to measure p_t. What quantity do you measure?

■

Suppose a particle-antiparticle pair is created outside the horizon. Since

$$p_t + \overline{p}_t = \text{constant},$$

and there was nothing before creation, the energy must be zero. However the energy can be nonzero over short times, because $\Delta E > \hbar/\Delta t$. The particle-antiparticle pair is "virtual." It lives only a short time.

Suppose a particle-antiparticle pair is created. The position is uncertain because $\Delta x > \hbar/\Delta p$. Suppose the particle is outside the horizon, and the antiparticle is inside the horizon.

Q: You set up a laboratory inside the horizon to measure p_t. What quantity do you measure? Reasoning?

■

Outside the horizon p_t is energy. Inside the horizon, $\overline{p}_t$ is radial momentum. It is OK for the energy of the particle to be positive and the radial momentum of the antiparticle to be negative: The particle-antiparticle pair can be real. The particle moves away from the BH.

Q: What should the spectrum of the particles be?

Q: Can black holes emit electrons?

Spectrum of a black hole

Spectrum of a low-temperature black hole (kT $\ll m_e$)

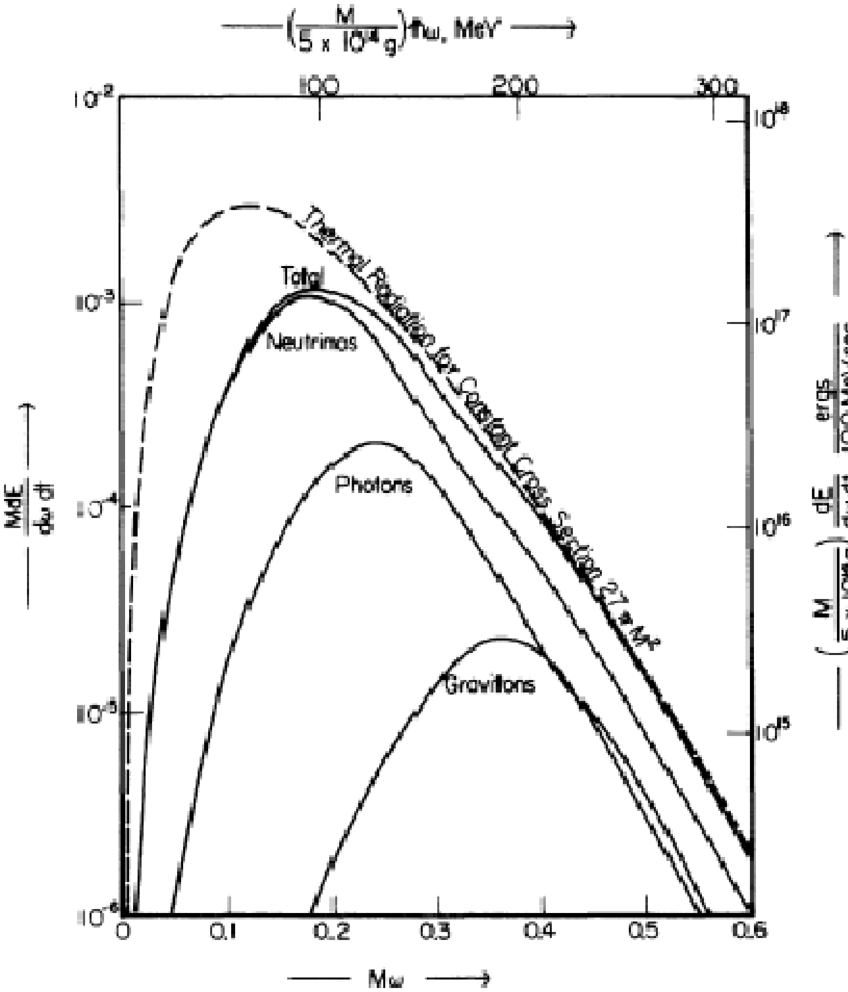

FIG. 1. Power spectra from a black hole, obtained by adding all angular modes for four kinds of neutrinos and for two polarization states (helicities) each of photons and gravitons. The lowest angular modes, l = s, dominate, but the 1 = S + 1 modes can be seen coming in with a small "bump" in the neutrino spectrum at $M\omega \approx 0.4$ and in the photon spectrum at $M\omega \approx 0.5$. The total power spectrum can be seen at high frequencies to approach that of a thermal body with a cross section of $27\pi M^2$,
but at low frequencies the spectrum drops below the Planck form as the cross section of the black hole is reduced. (Page, D, 1976, Phys Rev D, 13, 198)

Lifetime of a black hole

- **Black holes have a finite lifetime**

Black holes lose energy because of the emission of particles.
Q: Which has a higher temperature, a kg black hole or a 10-kg black hole?
Q: Which has a higher luminosity?
Q: Which has a shorter lifetime?