Orbits of particles with mass. Effective potential

Recall:
(1) The length\(^2\) of the 4-velocity is \(-1\).
(2) \(u_0\) is conserved because the metric is independent of time.
(3) \(u_3\) (in the \(\phi\) direction) is conserved because the metric is independent of \(\phi\).

From (1–3), we derived
\[
\frac{e^2 - 1}{2} = \frac{1}{2} \left( \frac{dr}{dt} \right)^2 + V_{\text{eff}}(r),
\]
where
\[
V_{\text{eff}}(r) = -\frac{M}{r} + \frac{\rho^2}{2r^2} - \frac{M\rho^2}{r^3}.
\]

Scale \(r\) and angular momentum \(\ell\) by dividing by \(M\).

\[
v_{\text{Eff}}[r, \ell] := -1 / r + \frac{\ell^2}{2 r^2} - \frac{\ell^2}{r^3};
\]
\[
v_{\text{EffNewton}}[r, \ell] := -1 / r + \frac{\ell^2}{2 r^2}.
\]

Caption: Effective potential for Einstein's theory (blue) and for Newton's theory (purple)

Orbit exists if \(\frac{e^2 - 1}{2} > V_{\text{eff}}(r)\) for some \(r\), and there are two turning points. The turning points are where \(\frac{e^2 - 1}{2} = V_{\text{eff}}(r)\).

- **Questions**

- For an orbit of a planet that just grazes the sun, what is the order of magnitude of \(r / M\)?
When orbits are not possible

Find extrema of $V_{\text{eff}}(r)$

\[
\text{Solve}\left[\frac{\partial}{\partial r} V_{\text{eff}}(r, l), \frac{\partial}{\partial r} V_{\text{eff}}(l, r) = 0, r\right]
\]

\[\left\{\left\{r \to \frac{1}{2} \left( l^2 - 1 \sqrt{-12 + 1^2} \right) \right\}, \left\{r \to \frac{1}{2} \left( l^2 + 1 \sqrt{-12 + 1^2} \right) \right\}\right\}
\]

If $l > 12^{1/2} M$, there are two extrema. If $l < 12^{1/2} M$, the extrema are imaginary.

\[\text{plotVEffective}[3.5, -0.053, \text{PlotRange} \to \{-0.06, 0.02\}]\]
How to find the orbit

Use a new variable $u = 1/r$, and substitute

$$\frac{dr}{d\tau} = \frac{du}{d\phi} \frac{d\phi}{d\tau} \frac{dr}{du} = \frac{du}{d\phi} \frac{1}{r^2} (r^2)$$

to get

$$\left( \frac{du}{d\phi} \right)^2 = \frac{\dot{r}^2 - 1}{r^2} + \frac{2M}{\mu} - u^2 + 2Mu^3.$$ 

Integrate to find $u(\phi)$. This needs to be done numerically.

To find $\phi(\tau)$, numerically integrate

$$d \tau = \frac{1}{r(\phi)^2} d \phi$$
Particles without mass. Light

For planets, we used

\[ u^\mu = \left( \frac{dt}{\tau^*}, \frac{dr}{\tau^*}, \frac{d\theta}{\tau^*}, \frac{d\phi}{\tau^*} \right) \]

We could just as well have used 4-momentum \( p^\mu = m u^\mu \).

For photons, that is not valid because proper time is 0.

Instead of proper time, use a parameter \( \lambda \). Then

\[ p^\mu = \left( \frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\phi}{d\lambda} \right) \]

We will solve a particular orbit and find out what \( \lambda \) is in that case.

Recall:
(1) The length of the 4-momentum \( p \) is 0.
(2) \( p_0 \) is conserved because the metric is independent of time. Define

\[ e = p_0 = \left( 1 - \frac{2M}{r} \right) \frac{dt}{d\lambda} \]

(3) \( p_3 \) (in the \( \phi \) direction) is conserved because the metric is independent of \( \phi \).

Define

\[ l = r^2 \frac{d\phi}{d\lambda} \]

From (1–3), we get

\[ -\left( 1 - \frac{2M}{r} \right)^{-1} e^2 + \left( 1 - \frac{2M}{r} \right)^{-1} \left( \frac{dr}{d\lambda} \right)^2 + \frac{l^2}{r^2} = 0. \]

(\( \frac{e}{l} \) \text{ is constant}) + \( W_{\text{eff}}(r) \),

where \( W_{\text{eff}}(r) = \frac{1}{r^2} \left( 1 - \frac{2M}{r} \right) \).

A photon is headed for \( r = b \) from the star. The parameter \( b \) is called the impact parameter. \( \sin \phi = b/r \), and \( \cos \phi \frac{d\phi}{d\lambda} = -\frac{b}{r^2} \frac{dr}{d\lambda} \).

Rewrite as

\[ \cos \phi \frac{d\phi}{d\lambda} = -\frac{b}{r^2} \frac{dr}{d\lambda} \]

As \( r \to \infty \), LHS \( \to l \) and RHS \( \to b \cdot e \). Therefore \( b = l/e \).

\[ \frac{1}{b^2} = \frac{1}{l^2} \left( \frac{dr}{d\lambda} \right)^2 + W_{\text{eff}}(r) \]

\[ \triangleleft \quad \triangleright \]
\[ w_{\text{Eff}}[r_] := -2 r^{-3} + r^{-2}; \]

\[
\text{plotWEffective}[b_\_., \text{OptionsPattern}[\text{PlotRange} \to \text{Automatic}]] :=
\]

\[
\text{Module}\left[\left\{r, \text{tp}, \text{rl}, \text{pr}\right\}, r = .; \text{tp} = r /. \text{Solve}\left[b^{-2} = w_{\text{Eff}}[r], r\right][2 ;; 3] \right\};
\]

\[
\text{pr} = \text{OptionValue}[\text{PlotRange}];
\]

\[
\text{rl} = \text{If}\left[\text{Length}\left[\text{Dimensions}\left[\text{pr}\right]\right] > 1, \text{pr}[1], \{2., 40\}\right];
\]

\[
\text{Plot}\left[w_{\text{Eff}}[r], \{r, \text{rl}[1], \text{rl}[2]\}, \text{BaseStyle} \to \{\text{FontFamily} \to \text{"Helvetica"}, \text{FontSize} \to \text{Medium}\},
\]

\[
\text{AxesLabel} \to \{\text{"r/M"}, \text{"W}_{\text{eff}}\}, \text{PlotRange} \to \text{OptionValue}[\text{PlotRange}],
\]

\[
\text{Epilog} \to \{\text{Dashed}, \text{If}[\text{Im}[\text{tp}[2]] \neq 0, \text{Line}(), \text{Line}\left[\{\{\text{tp}[2], b^{-2}\}, \{\text{rl}[2], b^{-2}\}\}\right]\}]\]

\]

\[
\text{plotWEffective}[10., \text{PlotRange} \to \{\{2., 40\}, \{0., .04\}\}]
\]

When \( b = 10 \, M \), photons go to in to \( r = 8.6 \, M \) and then back out again.

Find the peak of \( W_{\text{eff}} \).

\[
r / . \text{Solve}\left[D[w_{\text{Eff}}[r], r] = 0, r\right][1]
\]

\[
3
\]

Solve for \( b \)

\[
b = \text{Sqrt}[1 / w_{\text{Eff}}[\%]]
\]

\[
3 \sqrt{3}
\]
If $b < 27^{1/2} M$, photons go toward the mass and never go back out.

If the sun were a point mass, then the critical impact parameter for capture is

\[ m_{\text{Sun}} = 1.48 \text{ "km"}; \sqrt{27} \cdot m_{\text{Sun}} \]

7.69031 km
What is the parameter $\lambda$? Path of a radial light ray

\[ e = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda} \quad \text{and} \quad l = r^2 \frac{d\phi}{d\lambda}. \] What is $\lambda$?

We have differential equations for $\frac{dr}{d\lambda}$ and $\frac{d\phi}{d\lambda}$, which we can solve. Do the simple case of an almost radial light ray, for which $l \ll 1$. In that case \(e^2 = \left(\frac{\dot{r}}{\dot{\lambda}}\right)^2 = \frac{1}{r^2} \left(\frac{d\phi}{d\lambda}\right)^2 + W_{\text{eff}}(r)\) becomes

\[ e^2 = \left(\frac{d\phi}{d\lambda}\right)^2. \]

The solution is

\[ r = e \lambda. \]

Surprise: the parameter $\lambda$ is not time. For a radial light ray, the parameter $\lambda$ is the radial coordinate divided by the energy at $\infty$. (Recall $e$ is the energy at $r \to \infty$.)

Calculate the coordinate time.

\[ e = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda} \]
\[ = \left(1 - \frac{2M}{r}\right) \frac{dt}{dr} \frac{dr}{d\lambda}. \]

For radial paths, $e^2 = \left(\frac{dr}{d\lambda}\right)^2$. Substitute to get

\[ dt = \pm d r \left(1 - \frac{2M}{r}\right)^{-1} \]
\[ = \pm d r \left(1 + \frac{2M}{r-2M}\right) \]

Use $+$ for outgoing paths and $-$ for incoming paths. Substitute $r = e \lambda$ to get

\[ \Delta t = e \left(\lambda_2 - \lambda_1 + 2M \log \frac{\lambda_2 - 2M/e}{\lambda_1 - 2M/e}\right). \]

If the energy of the photon is bigger, the parameter $\lambda$ changes more slowly as $r$ and $t$ change. However the path $r(t)$ is independent of energy.
New Slide

Calculate the maximum orbital speed of a planet in circular orbit around a point mass.