Practice Midterm Exam #3

Total points = 25. Show all of your work!

1. [16 points] A planet (mass m) is in a stable circular orbit of radius r around a massive star (mass M with M >> m). The gravitational force of attraction is, of course, $F(r) = -GMm/r^2 \hat{r}$

(a) [2 points] What is the angular momentum of the orbit?

(b) [2 points] What is the Kinetic Energy?

(c) [2 points] What is the Potential Energy?

(d) [2 points] What is the Total Energy?

Now, by some strange aberration of the Theory of General Relativity, the mass of the star suddenly <u>increases</u> by a factor of 2.

(e) [2 points] What now is the Total Energy of the orbit?

Calculate/describe some features of the new orbit:-

(f) [2 points] Is it bound or unbound?

(g) [2 points] Is it a circle, ellipse, parabola, or hyperbola?

(h) [2 points] What is the eccentricity of the new orbit?

Note: There is another question on the next page!

2. [9 points] Astronauts are visiting the small asteroid Zarg, and want to know what is inside it. They drill a hole all the way through the asteroid, and then drop in a small probe equipped with an accelerometer. The probe oscillates back and forth between the starting point and the other side of Zarg, and sends acceleration data back to the astronauts. The data indicate that the acceleration of the probe can be written as $a = -Cr^2\hat{r}$ where C is a constant and r is the distance from the center of Zarg.

(a) [6 points] Assuming that Zarg is spherically symmetric, use Gauss' Law to find its density as a function of distance from the center, $\rho(r)$.

(b) [3 points] Use the Shell Theorem to check your answer. (This means, use your answer for $\rho(r)$ to calculate the acceleration of the probe as a function of r.)

Useful formulae and constants are on the back of this page.

PHY 321 Formulae and Constants

Gravity: $\vec{F} = -G\dfrac{Mm}{r^2}\hat{e}_r$ $\quad U(r) = -G\dfrac{Mm}{r}$ $\quad \vec{g} = \dfrac{\vec{F}}{m}$ $\quad \Phi = \dfrac{U}{m}$ $\quad \vec{g} = -\nabla\Phi$

Gauss' Law: $\int\vec{g}.d\vec{a} = -4\pi G\int\rho dV \implies \nabla\cdot\vec{g} = -4\pi G\rho$ Know the Shell Theorem!

Calculus in spherical coordinates: $\nabla\psi = \hat{e}_r\dfrac{\partial\psi}{\partial r} + \hat{e}_\theta\dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} + \hat{e}_\phi\dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial\phi}$

$$\nabla\bullet\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Central force motion: reduced mass $\mu = \dfrac{Mm}{M+m}$

$$\vec{L} = \vec{r}\times\vec{p} \qquad l = \left|\vec{L}\right| = \mu r^2\dot{\theta} \qquad K = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \tfrac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2}$$

$$E = K + U = \tfrac{1}{2}mv^2 + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + U(r) = \tfrac{1}{2}\mu\dot{r}^2 + V(r)$$

Orbits in gravitational field (Kepler's problem): $U(r) = -G\dfrac{Mm}{r} = \dfrac{-k}{r}$

$$\frac{\alpha}{r} = 1 + \varepsilon\cos\theta \qquad \alpha = \frac{l^2}{\mu k} \qquad \varepsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} \qquad a = \frac{k}{2|E|} = \frac{\alpha}{1-\varepsilon^2}$$

$\varepsilon = 0$: circle $\quad 0 < \varepsilon < 1$: ellipse $\quad \varepsilon = 1$: parabola $\quad \varepsilon > 1$: hyperbola

Elliptical orbits: $<K> = \dfrac{-1}{2}<U>$ $\quad E = \dfrac{-k}{2a}$ $\quad$ Kepler's 3rd Law: $\tau^2 = 4\pi^2\dfrac{\mu}{k}a^3$

Astronomical Data: $G = 6.67\times10^{-11}\ \mathrm{Nm^2/kg^2}$