QMII-1. Consider two kets and their corresponding column vectors:

\[
|\Psi\rangle = \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} \quad |\phi\rangle = \begin{pmatrix} 1 \\ 1 \\ -\sqrt{2} \end{pmatrix}
\]

Are these two state orthogonal? Is \( \langle \psi | \phi \rangle = 0 \)?
A) Yes \hspace{1cm} B) No

Answer: A

Are these states normalized? A) Yes \hspace{1cm} B) No

Answer: B (each state has a norm of 2)
QMII-2. In spin space, the basis states (eigenstates of $S^2$, $S_z$) are orthogonal: $\langle \uparrow | \downarrow \rangle = 0$.

Are the following matrix elements zero or non-zero?
$\langle \uparrow | S^2 | \downarrow \rangle$  $\langle \uparrow | S_z | \downarrow \rangle$

A) Both are zero  B) Neither are zero  
C) The first is zero; second is non-zero  
D) The first is non-zero; second is zero

Answer: A. Since the kets on the right are eigenstates of both operators, the eigenvalues can be pulled outside, and one is left with $\langle \uparrow | \downarrow \rangle = 0$. 
QMII-3. A spin ½ particle in the spin state $|\chi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$. A measurement of $S_z$ is made. What is the probability that the value of $S_z$ will be $+\hbar/2$?

A) $\left| \langle \uparrow | S_z | \chi \rangle \right|^2$

B) $\left| \langle \uparrow | \chi \rangle \right|^2$

C) $\left| \langle \chi | S_z | \chi \rangle \right|^2$

D) $\left| \langle \uparrow | S_z | \uparrow \rangle \right|^2$

E) None of these

Answer: B. Of course, this is also equal to $|a|^2$. 
QMII-4. The raising operator operating on the up and down spin states:
\( S_+ \left| \downarrow \right\rangle = \hat{h} \left| \uparrow \right\rangle, \quad S_+ \left| \uparrow \right\rangle = 0 \)
What is the matrix form of the \( S_+ \)?

A) \( \hat{h} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)
B) \( \hat{h} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \)
C) \( \hat{h} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \)
D) \( \hat{h} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \)
E) None of these.

Answer: B.

QMII-5. Is the raising operator \( S_+ \) Hermitian?
A) Yes, always \hspace{1cm} B) No, never \hspace{1cm} C) sometimes

Answer: B. The Hermitian conjugate of \( S_+ \) is \( S_- \).
QMII-6. Consider the matrix equation \[
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
\end{pmatrix} = \lambda
\begin{pmatrix}
a \\
b \\
\end{pmatrix}.
\]
This is equivalent to

A) \[
\begin{pmatrix}
-\lambda & 1 \\
1 & -\lambda \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
\end{pmatrix} = 0
\]

B) \[
\begin{pmatrix}
0 & 1-\lambda \\
1-\lambda & 0 \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
\end{pmatrix} = 0
\]

C) \[
\begin{pmatrix}
1-\lambda & 0 \\
0 & 1-\lambda \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
\end{pmatrix} = 0
\]

D) \[
\begin{pmatrix}
-\lambda & 1-\lambda \\
1-\lambda & -\lambda \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
\end{pmatrix} = 0
\]

E) None of these

Answer: A. (If you have trouble seeing this, insert the identity matrix between \(\lambda\) and the column vector on the right-hand side of the equation.)
QMII-7. Suppose a spin \( \frac{1}{2} \) particle is in the spin state \( |\chi\rangle = |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), the \(+\hbar/2\) eigenstate of \( \hat{S}_z \). Suppose we measure \( \hat{S}_x \) and then immediately measure \( \hat{S}_z \). What is the probability that the second measurement \( \hat{S}_z \) will leave the particle in the \( \hat{S}_z = \) down state: \( |\chi\rangle = |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)?

A) zero  B) non-zero

Answer: B. \( \hat{S}_x \) and \( \hat{S}_z \) are incompatible, so the measurement of \( \hat{S}_x \) will change the state to something that is not an eigenstate of \( \hat{S}_z \). The new state (which is an eigenstate of \( \hat{S}_x \)) will have nonzero components of both \( |\uparrow\rangle_z \) and \( |\downarrow\rangle_z \).
QMII-8. A quantum system consists of two particles, one of spin \( \frac{1}{2} \), and the other with spin \( \frac{3}{2} \). What is the dimension of the spin Hilbert space for this system?

A) \( \frac{3}{4} \)  
B) \( \frac{15}{4} \)  
C) 2  
D) 8  
E) I don’t know

Answer: D. Each spin Hilbert space has dimension \( (2s+1) \). So the tensor product space has dimension \( 2 \times 4 = 8 \).
QMII-9. Scandium has one electron in the 3d shell. If we measure the z-component of that electron’s total angular momentum, how many possible values might we get?

A) 2       B) 5       C) 6       D) 10       E) 12

Answer: C. This is a bit tricky. In the d shell, $l=2$, and the electron has $s=1/2$. So the total $j$ can be 5/2 or 3/2. Hence $m_j$ can be any of the six values with integer spacing between -5/2 and 5/2.
QMII-10. Consider Scandium again. If we measure $S^2$ of the 3d electron, what possible values might we get?

A) $\pm \frac{1}{2}\hbar$  
B) $\frac{1}{2}\hbar$ only  
C) $\frac{3}{4}\hbar^2$ only  
D) $\frac{3}{4}\hbar^2$ or $\frac{15}{4}\hbar^2$  
E) None of these

Answer: C. $s=1/2$ for an electron.
QMII-11. Consider Scandium again. If we measure $J^2$ ($J$ is the total angular momentum) of the 3d electron, what possible values might we get?

A) $\frac{3}{4} \hbar^2$  B) $\frac{15}{4} \hbar^2$  C) $\frac{35}{4} \hbar^2$  D) All of these  E) B and C only

Answer: E. In the answer to problem 9, we said that $j$ can be $5/2$ or $3/2$. 
QMII-12. Consider Scandium again. If we don’t know anything about the outermost electron other than that it is in a 3d orbital, what is the probability that a measurement of $J^2$ will produce the result $\frac{15}{4}\hbar^2$?

A) 1/2    B) 2/5    C) 3/7    D) 2/3
E) Impossible to compute without table of Clebsch-Gordan coefficients.

Answer: B. There are 4 states with $j=3/2$, and 6 states with $j=5/2$. If they are all equally probable, then the answer is $4/(4+6) = 2/5$. 
QMII-13. Which of the following two-electron quantum states satisfies the requirements of the Spin-Statistics Theorem? (For this problem, my notation for the spin states will be $|\chi_{s}^{m_{s}}\rangle$ rather than $|s,m_{s}\rangle$.

A) $|\Phi_{A}^{(1)}\Phi_{B}^{(2)}\rangle |\chi_{0}^{0}\rangle$

B) $\frac{1}{\sqrt{2}} \left( |\Phi_{A}^{(1)}\Phi_{B}^{(2)}\rangle - |\Phi_{B}^{(1)}\Phi_{A}^{(2)}\rangle \right) |\chi_{0}^{0}\rangle$

C) $\frac{1}{\sqrt{2}} \left( |\Phi_{A}^{(1)}\Phi_{B}^{(2)}\rangle + |\Phi_{B}^{(1)}\Phi_{A}^{(2)}\rangle \right) |\chi_{0}^{0}\rangle$

D) $\frac{1}{\sqrt{2}} \left( |\Phi_{A}^{(1)}\Phi_{B}^{(2)}\rangle + |\Phi_{B}^{(1)}\Phi_{A}^{(2)}\rangle \right) |\chi_{1}^{0}\rangle$

E) Both B and D

Answer: C. Electrons are Fermions, hence the state must be antisymmetric under exchange of the two particles. The $s=0$ spin state is antisymmetric under exchange, while the $s=1$ spin state is symmetric.
QMII-14. Consider two identical spin-1 particles. We want to find eigenstates of the total spin \( \vec{S} = \vec{S}_1 + \vec{S}_2 \). Which one of the following statements is correct? (I have omitted all tensor product symbols.)

A) \(|s = 1, m_s = 1\rangle = \frac{1}{\sqrt{2}} \left( |1,1\rangle^{(1)} |0,0\rangle^{(2)} + |1,0\rangle^{(1)} |1,1\rangle^{(2)} \right)\)

B) \(|s = 1, m_s = 1\rangle = \frac{1}{\sqrt{2}} \left( |1,1\rangle^{(1)} |1,0\rangle^{(2)} - |1,0\rangle^{(1)} |1,1\rangle^{(2)} \right)\)

C) \(|s = 1, m_s = 1\rangle = \frac{1}{\sqrt{2}} \left( |1,1\rangle^{(1)} |0,0\rangle^{(2)} + |0,0\rangle^{(1)} |1,1\rangle^{(2)} \right)\)

D) \(|s = 1, m_s = 1\rangle = \frac{1}{\sqrt{2}} \left( |1,1\rangle^{(1)} |1,1\rangle^{(2)} - |1,1\rangle^{(1)} |1,1\rangle^{(2)} \right)\)

E) Help, I need my Table of Clebsch-Gordan coefficients!

Answer: B. You can rule out C because one particle there has spin 0. You can rule out D because it would give \( m_s = 0 \). The state shown on the right-hand side in A is actually \(|s = 2, m_s = 1\rangle\), which you can show by applying the lowering operator to the state \(|s = 2, m_s = 2\rangle\).
QMII-15. Consider two identical spin-1 bosons. Which of the following two-electron quantum states satisfies the requirements of the Spin-Statistics Theorem? (For this problem, my notation for the spin states will be $|\chi_{s}^{m_s}\rangle$ rather than $|s,m_s\rangle$.

A) $\frac{1}{\sqrt{2}}\left(|\Phi_{A}^{(1)}\rangle|\Phi_{B}^{(2)}\rangle + |\Phi_{B}^{(1)}\rangle|\Phi_{A}^{(2)}\rangle\right)|\chi_{2}\rangle$

B) $\frac{1}{\sqrt{2}}\left(|\Phi_{A}^{(1)}\rangle|\Phi_{B}^{(2)}\rangle - |\Phi_{B}^{(1)}\rangle|\Phi_{A}^{(2)}\rangle\right)|\chi_{1}\rangle$

C) $\frac{1}{\sqrt{2}}\left(|\Phi_{A}^{(1)}\rangle|\Phi_{B}^{(2)}\rangle + |\Phi_{B}^{(1)}\rangle|\Phi_{A}^{(2)}\rangle\right)|\chi_{0}\rangle$

D) $\frac{1}{\sqrt{2}}\left(|\Phi_{A}^{(1)}\rangle|\Phi_{B}^{(2)}\rangle - |\Phi_{B}^{(1)}\rangle|\Phi_{A}^{(2)}\rangle\right)|\chi_{1}\rangle$

E) Both A and D

Answer: E. Since the particles are bosons, the state must be symmetric under exchange of the two particles. The states in the highest ladder ($s=2$ for this problem) are always symmetric under exchange. From the previous problem, you know that the states in the $s=1$ ladder are antisymmetric under exchange. All the states in a given spin ladder have the same symmetry under exchange.