Einstein's field equations
—4 Mar 2010

Einstein's field equation is

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}.$$

$G_{\mu\nu}$ describes the curvature of space. It involves derivatives of $g_{\mu\nu}$.

G is Newton's gravitational constant.

$T_{\mu\nu}$ is the stress-energy tensor. It describes the energy density, pressure, and stress present.

E's field equation says energy density, pressure, and stress causes curvature.

- Outline for the next month
  - Equivalence Principle shows Einstein how to incorporate gravity and special relativity. The happiest thought. (Today)
  - Experimental tests of the equivalence principle. (After spring break)
  - Following Einstein's path to E's field equations
  - Mathematics of curvature
  - Bianchi's identity
  - Discovery of E's field equation. November 1915.

Equivalence principle
Einstein's happiest thought

Pais, A., 1982, *Subtle is the Lord*, *Oxford*, p. 178, from a paper that Einstein wrote in 1907 but never published (the Morgan paper):

"Then there occurred to me the 'glücklichste Gedanke meines Lebens,' the happiest thought of my life, in the following form. The gravitational field has only a relative existence in a way similar to the electric field generated by magnetoelectric induction. *Because for an observer falling freely from the roof of a house there exists—at least in his immediate surroundings— no gravitational field* [his italics]. Indeed, if the observer drops some bodies then these remain relative to him in a state of rest or of uniform motion, independent of their particular chemical or physical nature (in this consideration the air resistance is, of course, ignored). The observer therefore has the right to interpret his state as 'at rest.'

Because of this idea, the uncommonly peculiar experimental law that in the gravitational field all bodies fall with the same acceleration attained at once a deep physical meaning. Namely, if there were to exist just one single object that falls in the gravitational field in a way different from all others, then with its help the observer could realize that he is in a gravitational field and is falling in it. If such an object does not exist, however—as experience has shown with great accuracy—then the observer lacks any objective means of perceiving himself as falling in a gravitational field. Rather he has the right to consider his state as one of rest and his environment as field-free relative to gravitation.

The experimentally known matter independence of the acceleration of fall is therefore a powerful argument for the fact that the relativity postulate has to be extended to coordinate systems which, relative to each other, are in nonuniform motion."

Q: E said he realized how to treat gravity and keep the victories of special relativity. What is his method?

Q: What did E says is an experimental test of the equivalence principle?

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Motion of a particle

Method:
1. Write the equation of motion without gravity.
2. Transform the equation to an accelerated frame. This incorporates gravity in every respect.

Painter's frame

Consider the frame of the painter falling off the roof of the house. The painter feels no gravity. The coordinates of a freely falling paint particle are ξ^μ. The metric is

$$d s^2 = \eta_{\mu\nu}\, d\xi^\mu\, d\xi^\nu.$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Without gravity, the equation of motion is

$$\frac{d^2 \xi^\mu}{d\tau^2} = 0$$

or in terms of the 4-velocity u^μ,

$$\frac{d u^\mu}{d\tau} = 0$$

Homeowner's frame

In the homeowner's frame, the coordinates of the paint particle is x^μ. The transformation to the painter's frame is

$$d\xi^\mu = \frac{\partial \xi^\mu}{\partial x^\nu}\, d x^\nu.$$

Q: Suppose the painter has just started to fall. What physics goes into $\frac{\partial \xi^\mu}{\partial x^\nu}$?

Q: Suppose the painter is falling in the z direction at speed v. What physics goes into $\frac{\partial \xi^\mu}{\partial x^\nu}$?

Q: Specifying $\frac{\partial \xi^\mu}{\partial x^\nu}$ does not allow every possible transformation of coordinates. What is a transformation that cannot be specified with $\frac{\partial \xi^\mu}{\partial x^\nu}$?

Because the length of a vector is independent of frame,

$$\begin{aligned} d s^2 &= \eta_{\mu\nu}\, d\xi^\mu\, d\xi^\nu . \\ &= \eta_{\mu\nu} \frac{\partial \xi^\mu}{\partial x^\alpha} \frac{\partial \xi^\nu}{\partial x^\beta}\, d x^\alpha\, d x^\beta \end{aligned}$$

Therefore in the homeowner's frame, the metric tensor

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial\xi^\mu}{\partial x^\alpha} \frac{\partial\xi^\nu}{\partial x^\beta}$$

The equation of motion:

$$0 = \frac{d^2\,\xi^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial\xi^\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau}\right)$$

$$= \frac{\partial\xi^\mu}{\partial x^\nu} \frac{d^2 x^\nu}{d\tau^2} + \frac{\partial^2\xi^\mu}{\partial x^\nu\,\partial x^\alpha} \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau}$$

Multiply by $\frac{\partial x^\lambda}{\partial\xi^\mu}$ and sum.

Note the transformation followed by the reverse is the identity.

$$\frac{\partial x^\lambda}{\partial\xi^\mu} \frac{\partial\xi^\mu}{\partial x^\nu} = \delta^\lambda_\nu.$$

The equation of motion in the homeowner's frame is

$$\frac{d^2 x^\lambda}{d\tau^2} + \left(\frac{\partial x^\lambda}{\partial\xi^\mu} \frac{\partial^2\xi^\mu}{\partial x^\nu\,\partial x^\alpha}\right) \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} = 0$$

The term is parenthesis is called the Christoffel symbol $\Gamma^\lambda_{\nu\alpha}$.

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\nu\alpha} \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} = 0$$

or

$$\frac{du^\lambda}{d\tau} + \Gamma^\lambda_{\nu\alpha}\, u^\nu\, u^\alpha = 0$$

Q: What have we picked up when the frame is not freely falling, when there is gravity?

With some work,

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2}\, g^{\nu\sigma}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)$$

Notation: Comma means partial derivative $g_{\mu\nu,\lambda} \equiv \frac{\partial g_{\mu\nu}}{\partial x^\lambda}$.

With gravity, the acceleration does change. The "force" depends on derivatives of the metric.

Q: Why must the force depend on derivatives of the metric and not only on the metric itself? Give an example that illustrates.

Newtonian limit

Consider the case of a ball and the stationary earth. Nonrelativistic limit $v \ll 1$: The 4-velocity

$$\frac{dx}{d\tau} = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau}\right) \approx (1, 0, 0, 0).$$

The equation of motion

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\nu\alpha} \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} = 0$$

is then

$$\frac{d^2 x^1}{d\tau^2} + \Gamma^1_{\nu\alpha} \frac{dx^\nu}{d\tau} \frac{dx^\alpha}{d\tau} = 0.$$

Since $\frac{dx^1}{d\tau}$ is of order v, whereas $\frac{dx^0}{d\tau}$ is of order 1, the biggest term involves Γ^λ_{00}

$$\Gamma^\sigma_{00} = \frac{1}{2} g^{\nu\sigma}\left(g_{0\,\nu,0} + g_{0\,\nu,0} - g_{00,\nu}\right)$$

Since earth is stationary, time derivatives of the metric are zero.

$$\Gamma^\sigma_{00} = -\frac{1}{2} g^{\nu\sigma} g_{00,\nu}$$

Assume the metric is

$$\begin{pmatrix} -1 - h_{00} & 0 & 0 & 0 \\ 0 & 1 - h_{11} & 0 & 0 \\ 0 & 0 & 1 - h_{22} & 0 \\ 0 & 0 & 0 & 1 - h_{33} \end{pmatrix}$$

$$\Gamma^0_{00} = -\frac{1}{2} g^{\nu 0} g_{00,\nu} = 0$$

$$\Gamma^1_{00} = -\frac{1}{2} g^{11} g_{00,1} = \frac{1}{2} \frac{\partial h_{11}}{\partial x^1}$$

The equation of motion is

$$\frac{d^2 x^1}{dt^2} + \frac{1}{2} \frac{\partial h_{11}}{\partial x^1} = 0.$$

I know that

$$\frac{d^2 \overline{x}}{dt^2} = -\text{grad}\,\phi$$

Therefore $h_{11} = 2\,\phi$. In the nonrelativistic limit, the x-x term in the metric is

$$g_{11} = 1 - 2\,\phi.$$

The 1 comes from the Minkowski metric. Gravity enters as twice the potential energy.

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Momentum conservation

We used Noether's theorem: If the metric does not change with a translation in the ν-th coordinate, then p_ν is conserved. We prove it.

http://owpdb.mfo.de/detail?photo_id=9267
Amalie Noether, March 23, 1882 - April 14, 1935.

We will show that

$$\frac{d\,u_\nu}{d\tau} - \frac{1}{2}\,g_{\alpha\beta,\nu}\,u^\alpha\,u^\beta = 0.$$

Then the proof is easy. If the metric does not change with a translation in the ν coordinate, then $g_{\alpha\beta,\nu} = 0$ and $\frac{d\,u_\nu}{d\tau} = 0$.

Proof:

$$\frac{du_\nu}{d\tau} = \frac{d\,g_{\nu\mu}\,u^\mu}{d\tau} = g_{\nu\mu}\,\frac{du^\mu}{d\tau} + u^\mu\,g_{\nu\mu,\sigma}\,\frac{dx^\sigma}{d\tau}$$

Q: Why does it mean to compute $\frac{d}{d\tau}\,g_{\nu\mu}$? Why does $g_{\nu\mu}$ change?

$$\begin{aligned}
\frac{du_\nu}{d\tau} &= \frac{d\,g_{\nu\mu}\,u^\mu}{d\tau} = g_{\nu\mu}\,\frac{du^\mu}{d\tau} + u^\mu\,g_{\nu\mu,\sigma}\,\frac{dx^\sigma}{d\tau} \\
&= -g_{\nu\mu}\,\Gamma^\mu_{\alpha\beta}\,u^\alpha\,u^\beta + g_{\nu\mu,\sigma}\,u^\mu\,u^\sigma \\
&= -\frac{1}{2}\,g_{\nu\mu}\,g^{\mu\gamma}\left(g_{\alpha\gamma,\beta} + g_{\gamma\beta,\alpha} - g_{\beta\alpha,\gamma}\right)u^\alpha\,u^\beta + g_{\nu\mu,\sigma}\,u^\mu\,u^\sigma \\
&= -\frac{1}{2}\left(g_{\alpha\nu,\beta} + g_{\nu\beta,\alpha} - g_{\beta\alpha,\nu}\right)u^\alpha\,u^\beta + g_{\nu\mu,\sigma}\,u^\mu\,u^\sigma
\end{aligned}$$

Since α, β, μ, and σ are indices that are summed,

$$\begin{aligned}
\frac{du_\nu}{d\tau} &= -\frac{1}{2}\left(g_{\alpha\nu,\beta} - g_{\nu\beta,\alpha} - g_{\beta\alpha,\nu}\right)u^\alpha\,u^\beta \\
&= \frac{1}{2}\,g_{\beta\alpha,\nu}\,u^\alpha\,u^\beta
\end{aligned}$$

Derivation of $\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma}\left(g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)$

- **A few identities**

Proving this equation is mostly algebra, but you will need some identities.

Q: What is $\frac{\partial \xi^{\beta}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial \xi^{\mu}}$?

- **A few identities**

Q: What is $\frac{\partial \xi^{\beta}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial \xi^{\mu}}$?

The equation $\frac{\partial \xi^{\beta}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial \xi^{\mu}}$ is the transformation of the μ component of ξ in the painter's frame to the homeowner's frame and back again, looking at the β component. This is 1 if $\mu = \beta$ and 0 if $\mu \neq \beta$. Therefore

$$\frac{\partial \xi^{\beta}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial \xi^{\mu}} = \delta^{\beta}_{\mu}$$

For homework: Prove $g^{\nu\sigma} g_{\lambda\nu} = \delta^{\sigma}_{\lambda}$.

Q: What does this equation say in words?

- **Proof**

$\Gamma^{\lambda}_{\nu\alpha} \equiv \frac{\partial x^{\lambda}}{\partial \xi^{\mu}} \frac{\partial^2 \xi^{\mu}}{\partial x^{\nu} \partial x^{\alpha}}$

I am going to make $\frac{\partial x^{\lambda}}{\partial \xi^{\mu}}$ disappear. Multiply $\Gamma^{\lambda}_{\nu\alpha} \frac{\partial \xi^{\beta}}{\partial x^{\lambda}}$

$$\Gamma^{\lambda}_{\nu\alpha} \frac{d\xi^{\beta}}{dx^{\lambda}} = \frac{\partial \xi^{\beta}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial \xi^{\mu}} \frac{\partial^2 \xi^{\mu}}{\partial x^{\nu} \partial x^{\alpha}}$$
$$= \delta^{\beta}_{\mu} \frac{\partial^2 \xi^{\mu}}{\partial x^{\nu} \partial x^{\alpha}}$$
$$= \frac{\partial^2 \xi^{\beta}}{\partial x^{\nu} \partial x^{\alpha}}$$

Derivative of $g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}}$

$$g_{\mu\nu,\gamma} \equiv \frac{\partial g_{\mu\nu}}{\partial x^{\gamma}}$$
$$= \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 \xi^{\beta}}{\partial x^{\nu} \partial x^{\gamma}} + \eta_{\alpha\beta} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\lambda}}$$
$$= \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\nu\gamma} + \eta_{\alpha\beta} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\mu\gamma}$$
$$= g_{\mu\lambda} \Gamma^{\lambda}_{\nu\gamma} + g_{\lambda\nu} \Gamma^{\lambda}_{\mu\gamma}$$

Rotate indices to get

$$g_{\nu\gamma,\mu} = g_{\nu\lambda} \Gamma^{\lambda}_{\gamma\mu} + g_{\lambda\gamma} \Gamma^{\lambda}_{\nu\mu}$$

and

$$g_{\gamma\mu,\nu} = g_{\gamma\lambda} \Gamma^{\lambda}_{\mu\nu} + g_{\lambda\mu} \Gamma^{\lambda}_{\gamma\nu}.$$

Add terms with ν on g and subtract term with ν on the derivative to get

$$g_{\mu\nu,\gamma} + g_{\nu\gamma,\mu} - g_{\gamma\mu,\nu} = g_{\mu\lambda} \Gamma^{\lambda}_{\nu\gamma} + g_{\lambda\nu} \Gamma^{\lambda}_{\mu\gamma} + g_{\nu\lambda} \Gamma^{\lambda}_{\gamma\mu} + g_{\lambda\gamma} \Gamma^{\lambda}_{\nu\mu} - g_{\gamma\lambda} \Gamma^{\lambda}_{\mu\nu} - g_{\lambda\mu} \Gamma^{\lambda}_{\gamma\nu}$$
$$= g_{\lambda\nu} \Gamma^{\lambda}_{\mu\gamma} + g_{\nu\lambda} \Gamma^{\lambda}_{\gamma\mu}$$
$$= 2\, g_{\lambda\nu} \Gamma^{\lambda}_{\mu\gamma}$$

Multiply by $g^{\nu\sigma}$ to get for the RHS

$$2\, g^{\nu\sigma} g_{\lambda\nu} \Gamma^{\lambda}_{\mu\gamma}$$
$$= 2\, \delta^{\sigma}_{\lambda} \Gamma^{\lambda}_{\mu\gamma} = 2\, \Gamma^{\sigma}_{\mu\gamma}$$

At last,

$$\Gamma^{\sigma}_{\mu\gamma} = \frac{1}{2} g^{\nu\sigma} \left(g_{\mu\nu,\gamma} + g_{\nu\gamma,\mu} - g_{\gamma\mu,\nu} \right)$$