Robertson-Walker metric—1 Apr 2010

A 3-d space that is homogeneous and isotropic has a special choice of time. A choice of coordinates is

(t, r, θ, ϕ)

and the metric is

$$\boxed{ds^2 = -dt^2 + a(t)^2 \left[\frac{d r^2}{1-(r/r_0)^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]}$$

(r, θ, ϕ) is called the comoving coordinate. A galaxy stays at the same position; time changes.

r_0^2 can have any value, positive or negative

$a(t)$ is called the expansion parameter.

Friedman's equation is

$$\boxed{\left(\frac{1}{a}\ \frac{d a}{d t}\right)^2 - \frac{8\pi}{3} G \rho = -\frac{1}{r_0^2 a^2}}$$

We derived Fiedmann's equation except for the constant $-\frac{1}{r_0^2}$ on the RHS.

- Outline
  - Derivation of the Robertson-Walker metric
  - Derivation of Friedman's equation from the Robertson-Walker metric

From of the metric

Assumptions:
1) There is a time coordinate that is proper time.
2) At a given time, the space within a small bubble is isotropic.
3) At a given time, the space is homogeneous.

The metric is

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + A(t, \text{vector } r)\left(\mathrm{dr}^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right)$$

Q: What assumptions have gone into writing a metric of this form?

Since the space is homegeneous,

$$A(t, r_1)/A(t, r_2)$$

can depend on time and also on the distance between the points. It cannot depend on the location. Therefore

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + a(t)^2\, B(r)\left(\mathrm{dr}^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right)$$

Q: Have I eliminated the possibility of a curved space by grouping the dr, dθ, and dϕ terms together as $\left(\mathrm{dr}^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right)$, which is the case for flat, spherical coordinates?

By mapping the 3-d onto the surface of a 4-d symmetric space, we can show that the possible metrics are

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + a(t)^2\left(\frac{\mathrm{dr}^2}{1-\left(\frac{r}{r_0}\right)^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right),$$

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + a(t)^2\left(\frac{\mathrm{dr}^2}{1+\left(\frac{r}{r_0}\right)^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right),$$

and

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + a(t)^2\left(\mathrm{dr}^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right).$$

Q: Interpret the spatial parts of this metric.

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Derivation of Friedman's equation

The metric is

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{d\,r^2}{1-(r/r_0)^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Since the coordinate time is the same as proper time, in average the galaxies are at rest.
The stress-energy tensor

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

Q: Galaxies occur in clusters. If the motion of the galaxies is faster, does that change the stress-energy tensor?

Einstein's equation is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = -8\,\pi\, G\, T_{\mu\nu}$$

Plan: Calculate the curvature tensors and find conditions on $a(t)$ and r_0 that satisfy E's equations.

Rewrite

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{d\,r^2}{1-(r/r_0)^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

as

$$ds^2 = -dt^2 + a(t)^2 \left(\tilde{g}_{11}\, dx_1^2 + \tilde{g}_{22}\, dx_2^2 + \tilde{g}_{33}\, dx_3^2\right)$$

The derivatives of $\tilde{g}$ are not zero. At small r, $\tilde{g} = \begin{pmatrix} 1 & 0 & \square \\ 0 & 1 & \square \\ \square & \square & 1 \end{pmatrix}$

1. Easy one: We found (on 3/31) that the curvature scalar

$$R = 4\,\pi\, G\, T^{\alpha}{}_{\alpha}.$$

Here

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & a^2 P & 0 & 0 \\ 0 & 0 & a^2 P & 0 \\ 0 & 0 & 0 & a^2 P \end{pmatrix}$$

$$R = 4\,\pi\, G\left(-\rho + 3\, a^2\, P\right).$$

Derivation of Friedman's equation, part 2

2. Compute Christoffel symbols

a) Time part

$$\Gamma^0{}_{\alpha\beta} = \frac{1}{2} g^{00}\left(g_{0\,\alpha,\,\beta} + g_{0\,\beta,\,\alpha} - g_{\alpha\beta,\,0}\right)$$

$\alpha = 0$:

$$\Gamma^0{}_{0\,\beta} = \frac{1}{2} g^{00}\left(g_{00,\,\beta} + g_{0\,\beta,\,0} - g_{0\,\beta,\,0}\right) = \frac{1}{2} g^{00}\, g_{00,\,\beta} = 0,$$

because $g_{00} = -1$.

$\alpha = i \neq 0$ and $\beta = j \neq 0$:

$$\Gamma^0{}_{ij} = \frac{1}{2} g^{00}\left(g_{0\,i,\,j} + g_{0\,j,\,i} - g_{ij,\,0}\right)$$

The first two terms are zero because ???.

$$\Gamma^0{}_{ij} = -\frac{1}{2}\left(-g_{ij,\,0}\right) = a(t)\, \frac{da}{dt}\, \tilde{g}_{\mathrm{ij}}.$$

b) Space part

$$\Gamma^i{}_{\alpha\beta} = \frac{1}{2} g^{ii}\left(g_{\mathrm{i}\alpha,\,\beta} + g_{\mathrm{i}\beta,\,\alpha} - g_{\alpha\beta,\,i}\right)$$

$\alpha = 0$:

$$\Gamma^i{}_{0\,\beta} = \frac{1}{2} g^{ii}\left(g_{\mathrm{i}0,\,\beta} + g_{\mathrm{i}\beta,\,0} - g_{0\,\beta,\,i}\right)$$

The first term is zero. Second term is zero unless $\beta = i$, in which case it is $\frac{1}{2} g^{\mathrm{ii}}\, 2\, a\, \dot{a}\, \tilde{g}_{\mathrm{ii}} = \frac{\dot{a}}{a}$. The third term is zero, since $g_{00} = -1$.

$$\Gamma^i{}_{0\,\beta} = \frac{\dot{a}}{a}\, \delta^i_\beta.$$

$\alpha = i \neq 0$ and $\beta = j \neq 0$:

$$\Gamma^i{}_{\mathrm{jk}} = \frac{1}{2} g^{\mathrm{ii}}\left(g_{\mathrm{ij},\,k} + g_{\mathrm{ik},\,j} - g_{\mathrm{jk},\,i}\right)$$

involves space coordinates only.