In the space outside a spherical mass M at the origin, the distance ds between two nearby events is

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

1. (5 pts.) For the Schwartzschild metric, $g_{\theta\theta} = r^2$. Find $g^{\theta\theta}$. You must show your reasoning in sentences.

2. A photon moves radially towards a star of mass M. Its 4-momentum at ∞ is $p^\mu = (E, -E, 0, 0)$.

   (a) (5 pts.) Find its 4-momentum p^μ at r.

   (b) (5 pts.) Find the energy that a person would measure in a lab at r.

3. A person on earth determines the length of an earth year to be t_e and a stationary person far, far away determines the earth year to be t_∞. The mass of the sun $M_{\text{sun}} = 1.5\,\text{km}$. The mass of the earth $M_{\text{earth}} = 4.3 \times 10^{-6}\,\text{km}$. The distance between the earth and sun, an astronomical unit AU $= 1.5 \times 10^8\,\text{km}$. The radius of the earth $R_{\text{earth}} = 6.4 \times 10^3\,\text{km}$. The orbital speed of earth is 0.001.

   (a) (5 pt.) What are the effects that make t_e and t_∞ different?

   (b) (5 pt.) Compute $t_e/t_\infty - 1$.

4. We derived the equation of a mass in orbit around a star of mass M,

$$\frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{2r^2} - \frac{M}{r} - \frac{l^2 M}{r^3},$$

   where e and l are constants specific to the orbit.

   (a) (5 pts.) Show that the minimum angular momentum for an orbit where r goes in and turns around is $l = M\sqrt{12}$.

   (b) (5 pts.) For objects that are moving at $v = 0.001$ at $r = \infty$, find the cross section for capture by a black hole of mass M. (The cross section for capture is defined to be the area of a plane at $r = \infty$ for which objects moving perpendicular to the plane are captured.)