Orbits of particles with mass. Effective potential

Recall:

(1)The length^2 of the 4-velocity is -1.

(2) $u_0$is conserved because the metric is independent of time.

(3) u_3(in the ϕ direction) is conserved because the metric is independent of ϕ.

From (1–3), we derived

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{d\,r}{d\,\tau}\right)^2 + V_{\text{eff}}(r),$$

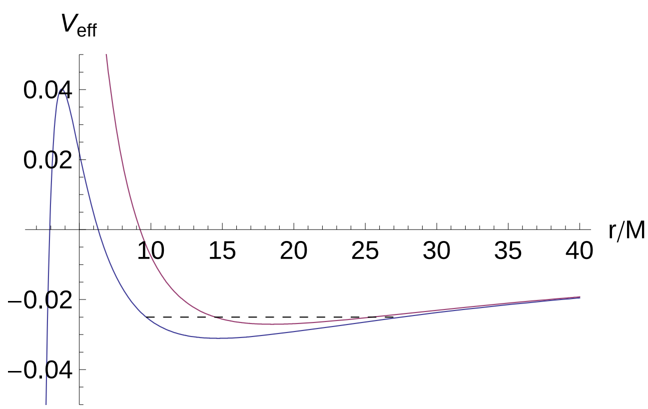
where $V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2\,r^2} - \frac{M\,l^2}{r^3}$

Scale r and angular momentum l by dividing by M.

```
vEff[r_, l_] := -1 / r + l^2/(2 r^2) - l^2/r^3; vEffNewton[r_, l_] := -1 / r + l^2/(2 r^2)
```

```
plotVEffective[l_, e_, OptionsPattern[PlotRange → Automatic]] :=
 Module[{r, tp, rl, pr}, r =.; tp = r /. Solve[e == vEff[r, l], r][[2 ;; 3]];
  pr = OptionValue[PlotRange];
  rl = If[Length@Dimensions@pr > 1, pr[[1]], {2, 40}]; Plot[{vEff[r, l], vEffNewton[r, l]},
   {r, rl[[1]], rl[[2]]}, BaseStyle → {FontFamily → "Helvetica", FontSize → Medium},
   AxesLabel → {"r/M", "V_eff"}, PlotRange → OptionValue[PlotRange],
   Epilog → {Dashed, If[Im[tp[[1]]] ≠ 0, Line[{}], Line[{#, e} & /@ tp]]}]
 ]
```

```
plotVEffective[4.3, -.025, PlotRange → {-.05, .05}]
```



Caption: Effective potential for Einstein's theory (blue) and for Newton's theory (purple)

Orbit exists if $\frac{e^2-1}{2} > V_{\text{eff}}(r)$ for some r, and there are two turning points. The turning points are where $\frac{e^2-1}{2} = V_{\text{eff}}(r)$.

- **Questions**

- **For an orbit of a planet that just grazes the sun, what is the order of magnitude of r/M?**

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When orbits are not possible

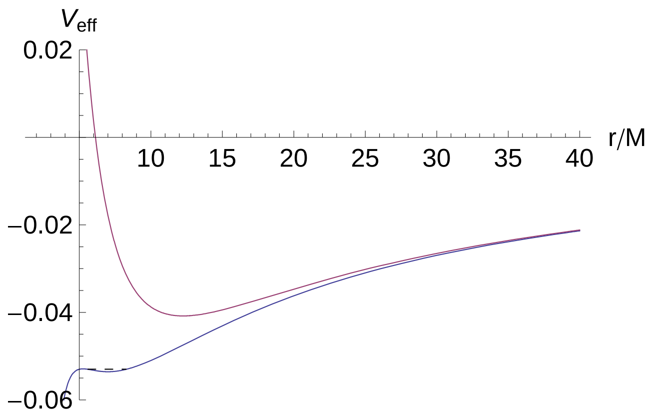
Find extrema of $V_{\text{eff}}(r)$

```
Solve[D[vEff[r, l], r] == 0, r]
```

$$\left\{\left\{r \to \frac{1}{2}\left(l^2 - l\sqrt{-12 + l^2}\right)\right\}, \left\{r \to \frac{1}{2}\left(l^2 + l\sqrt{-12 + l^2}\right)\right\}\right\}$$

If $l > 12^{1/2} M$, there are two extrema. If $l < 12^{1/2} M$, the extrema are imaginary.

```
plotVEffective[3.5, -.053, PlotRange → {-.06, .02}]
```



How to find the orbit

Use a new variable $u = 1/r$, and substitute $\frac{d\,r}{d\,\tau} = \frac{d\,u}{d\,\phi}\,\frac{d\,\phi}{d\,\tau}\,\frac{d\,r}{d\,u} = \frac{d\,u}{d\,\phi}\,\frac{l}{r^2}\left(-r^2\right)$ to get

$$\left(\frac{\mathrm{du}}{d\,\phi}\right)^2 = \frac{e^2-1}{l^2} + \frac{2\,M\,u}{l^2} - u^2 + 2\,M\,u^3.$$

Integrate to find $u(\phi)$. This needs to be done numerically.

To find $\phi(\tau)$, numerically integrate

$$d\,\tau = \frac{1}{l}\,r(\phi)^2\,d\,\phi$$

Particles without mass. Light

For planets, we used

$$u^\mu = \left(\frac{d\,t}{d\,\tau}, \frac{d\,r}{d\,\tau}, \frac{d\,\theta}{d\,\tau}, \frac{d\,\phi}{d\,\tau}\right).$$

We could just as well have used 4-momentum $p^\mu = m\,u^\mu$.

For photons, that is not valid because proper time is 0.

Instead of proper time, use a parameter λ. Then

$$p^\mu = \left(\frac{d\,t}{d\,\lambda}, \frac{d\,r}{d\,\lambda}, \frac{d\,\theta}{d\,\lambda}, \frac{d\,\phi}{d\,\lambda}\right).$$

We will solve a particular orbit and find out what λ is in that case.

Recall:

(1) The length of the 4-momentum p is 0.

(2) $p_0$is conserved because the metric is independent of time. Define

$$e = p_0 = \left(1 - \frac{2\,M}{r}\right)\frac{d\,t}{d\,\lambda}.$$

(3) p_3(in the ϕ direction) is conserved because the metric is independent of ϕ.

Define

$$l = r^2\,\frac{d\,\phi}{d\,\lambda}$$

From (1–3), we get

$$-\left(1 - \frac{2\,M}{r}\right)^{-1} e^2 + \left(1 - \frac{2\,M}{r}\right)^{-1}\left(\frac{d\,r}{d\,\lambda}\right)^2 + \frac{l^2}{r^2} = 0.$$

$$\left(\frac{e}{l}\right)^2 = \frac{1}{l^2}\left(\frac{d\,r}{d\,\lambda}\right)^2 + W_{\text{eff}}(r),$$

where $W_{\text{eff}}(r) = \frac{1}{r^2}\left(1 - \frac{2\,M}{r}\right)$.

A photon is headed for $r = b$ from the star. The parameter b is called the impact parameter. $\sin\phi = b/r$, and $\cos\phi\,\frac{d\,\phi}{d\,\lambda} = -\frac{b}{r^2}\,\frac{d\,r}{d\,\lambda}$.

Rewrite as

$$\cos\phi\;l = -b\,\frac{d\,r}{d\,\lambda}$$

As $r \to \infty$, LHS $\to l$ and RHS $\to b\,e$. Therefore $b = l/e$.

$$\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{d\,r}{d\,\lambda}\right)^2 + W_{\text{eff}}(r),$$

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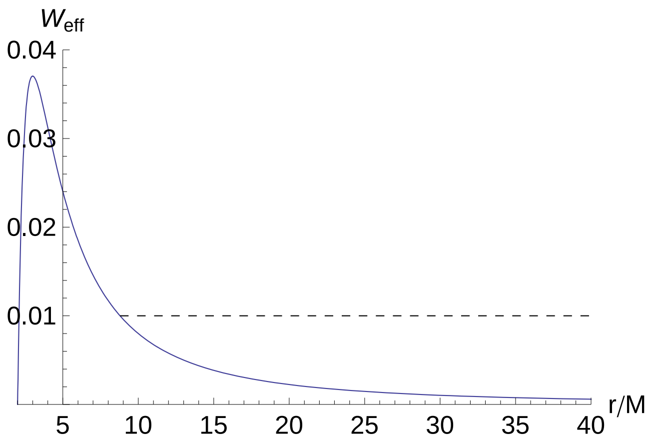
```
wEff[r_] := -2 r^-3 + r^-2;

plotWEffective[b_, OptionsPattern[PlotRange → Automatic]] :=
 Module[{r, tp, rl, pr}, r =.; tp = r /. Solve[b^-2 == wEff[r], r][[2 ;; 3]];
  pr = OptionValue[PlotRange];
  rl = If[Length@Dimensions@pr > 1, pr[[1]], {2, 40}];
  Plot[wEff[r], {r, rl[[1]], rl[[2]]}, BaseStyle → {FontFamily → "Helvetica", FontSize → Medium},
   AxesLabel → {"r/M", "W_eff"}, PlotRange → OptionValue[PlotRange],
   Epilog → {Dashed, If[Im[tp[[2]]] ≠ 0, Line[{}], Line[{{tp[[2]], b^-2}, {rl[[2]], b^-2}}]]}]
 ]

plotWEffective[10., PlotRange -> {{2, 40}, {0, .04}}]
```



When $b = 10\,M$, photons go to in to $r = 8.6\,M$ and then back out again.

Find the peak of W_{eff}.

```
r /. Solve[D[wEff[r], r] == 0, r][[1]]
```

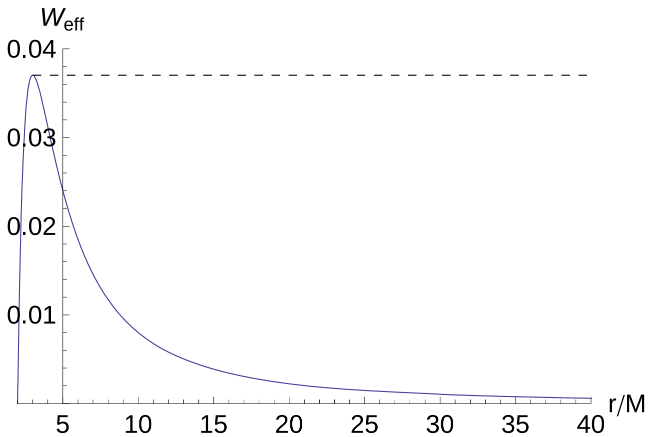
3

Solve for b

```
b = Sqrt[1 / wEff[%]]
```

$3\sqrt{3}$

```
plotWEffective[3 Sqrt[3], PlotRange -> {{2, 40}, {0, .04}}]
```



If $b < 27^{1/2} M$, photons go toward the mass and never go back out.

If the sun were a point mass, then the critical impact parameter for capture is

```
mSun = 1.48 "km"; Sqrt[27] mSun
```

```
7.69031 km
```

What is the parameter λ? Path of a radial light ray

$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}$ and $l = r^2 \frac{d\phi}{d\lambda}$. What is λ?

We have differential equations for $\frac{dr}{d\lambda}$ and $\frac{d\phi}{d\lambda}$, which we can solve. Do the simple case of an almost radial light ray, for which $l \ll 1$. In that case $\left(\frac{e}{l}\right)^2 = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{\text{eff}}(r)$ becomes

$$e^2 = \left(\frac{dr}{d\lambda}\right)^2.$$

The solution is

$$r = e\,\lambda.$$

Surprise: the parameter λ is not time. For a radial light ray, the parameter λ is the radial coordinate divided by the energy at ∞. (Recall e is the energy at $r \to \infty$.)

Calculate the coordinate time.

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}$$
$$= \left(1 - \frac{2M}{r}\right)\frac{dt}{dr}\frac{dr}{d\lambda}$$

For radial paths, $e^2 = \left(\frac{dr}{d\lambda}\right)^2$. Substitute to get

$$dt = \pm dr\left(1 - \frac{2M}{r}\right)^{-1}$$
$$= \pm dr\left(1 + \frac{2M}{r-2M}\right)$$

Use + for outgoing paths and – for incoming paths. Substitute $r = e\,\lambda$ to get

$$\Delta t = e\left(\lambda_2 - \lambda_1 + 2M \log \frac{\lambda_2 - 2M/e}{\lambda_1 - 2M/e}\right)$$

If the energy of the photon is bigger, the parameter λ changes more slowly as r and t change. However the path $r(t)$ is independent of energy.

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Calculate the maximum orbital speed of a planet in circular orbit around a point mass.