What is r_0^2?
—4 Feb 2010

The FRW metric is

$$ds^2 = -dt^2 + a(t)^2 \left[d\,r^2 \big/ \left(1 - (r/r_0)^2\right) + r^2 \left(d\,\theta^2 + \sin^2\theta\, d\,\phi^2\right)\right]$$

Galaxies are at fixed (r, θ, ϕ).

Let time be fixed. Consider the spatial part of the metric.

$$d\,r^2 \big/ \left(1 - (r/r_0)^2\right) + r^2 \left(d\,\theta^2 + \sin^2\theta\, d\,\phi^2\right)$$

Suppose $r_0^2 \to \infty$. What is the space?

Suppose $r_0^2 > 0$. What is the space?

The radial distance between $r = 0$ and r is

$$\int_0^r d\,x \big/ \left[1 - (x/r_0)^2\right]^{1/2} = r_0 \int_0^{r/r_0} \frac{d\,y}{\left(1-y^2\right)^{1/2}} = r_0 \sin^{-1}(r/r_0).$$

The circumferential distance is

$$r\,(\theta_2 - \theta_1)$$

This has a simple interpretation:

Map 3-d space into 2-d by supressing ϕ. The space is the 2-d surface of a sphere. This means the surface is the entire space. Above and below the surface is not part of the space.

I draw a slice through the sphere.

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fig[]
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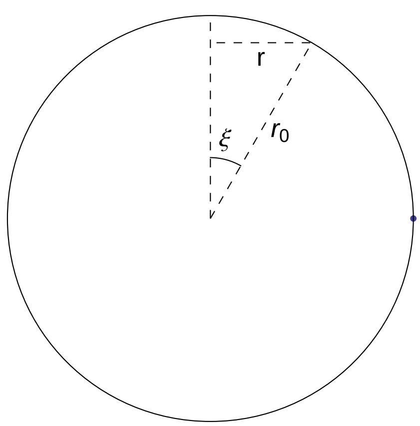

Define $\sin\xi = r/r_0$. $r_0\, d\,\xi = d\,r/\cos\xi = d\,r \big/ \left[1 - (r/r_0)^2\right]$. Therefore

$$ds^2 = r_0^2\, d\text{ latitude}^2 + r^2\, d\text{ longitude}^2.$$

Lesson: If $r_0^2 > 0$, then r_0 is the radius of curvature of the 3-d space.

As r increases, the radial distance increases. Beyond $r/r_0 = 1$, r decreases, but the radial distance keeps increasing.

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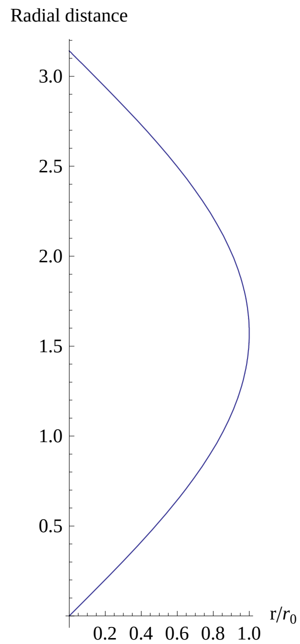
ParametricPlot[{Sin[x], x}, {x, 0, π},
 ImageSize → {300, 300}, AxesLabel → {"r/r0", "Radial distance"}]
```



If $r_0^2 < 0$, the space is like a saddle, and the space is infinite.

Hubble's Law

Consider the case $r^2 \ll \left| r_0{}^2 \right|$. The radial distance of a galaxy at r is

$$D(t) = a(t)\, r.$$

The galaxy moves away at speed

$$v = \frac{d}{d\,t} D(t) = \frac{1}{a} \frac{d\,a(t)}{d\,t}\, a\, r = \frac{1}{a} \frac{d\,a(t)}{d\,t} D.$$

Define Hubble's constant $H = \frac{1}{a} \frac{d\,a(t)}{d\,t}$. Then

$$v = H\, D$$

This is Hubble's Law.

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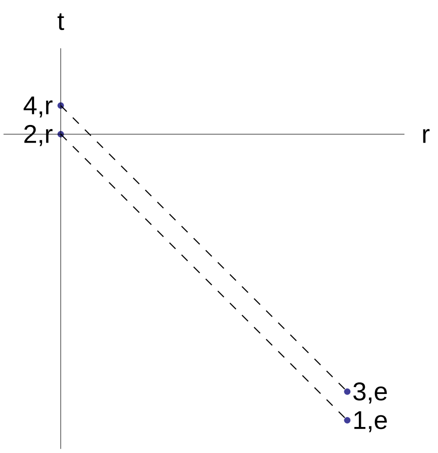
```
fig[] := Module[{dg = {1, -1}, mw = {0, 0}, ϵ = {0, .1}},
  ListPlot[{dg, mw, dg + ϵ, mw + ϵ}, AxesLabel → {"r", "t"}, AspectRatio → Automatic,
   ImageSize → 200, BaseStyle → {FontFamily → "Helvetica", FontSize → Medium}, Epilog →
    {Text["1,e", dg, {-1.3, 0}], Text["3,e", dg + ϵ, {-1.3, 0}], Text["2,r", mw, {1.5, 0}],
     Text["4,r", mw + ϵ, {1.5, 0}], Dashed, Line[{mw, dg}], Line[{mw + ϵ, dg + ϵ}]},
   PlotRange → {{-.2, 1.2}, {-1.1, .3}}, Ticks → None]]
```

Redshift derived from the metric

A distant galaxy emits some light with wavelength λ_1 when the expansion parameter was $a(t_1)$. We see the light after some time. What is the wavelength of the light that we see?

```
fig[]
```



This is not our usual space-time diagram, since galaxies are at fixed spatial points, even though they are moving.

Events:
1. Distant galaxy emits a crest.
3. Distant galaxy emits a second crest.
2. Astronomer receives first crest.
4. Astronomer receives second crest.

The metric is

$$ds^2 = -dt^2 + a(t)^2 \left[d\,r^2 \big/ \left(1 - (r/r_0)^2\right) + r^2\left(d\,\theta^2 + \sin^2\theta\, d\,\phi^2\right)\right].$$

Integrate between events 1 and 2. Because $\boxed{?}$,

$$\int_1^2 \frac{d\,t}{a(t)} = \int_1^2 \left[1 - (r/r_0)^2\right]^{-1/2} d\,r$$

Similarly for the second crest,

$$\int_3^4 \frac{d\,t}{a(t)} = \int_3^4 \left[1 - (r/r_0)^2\right]^{-1/2} d\,r.$$

Since $r_1 = r_3$ and $r_2 = r_4$,

$$\int_1^2 \frac{d\,t}{a(t)} = \int_3^4 \frac{d\,t}{a(t)}.$$

$$\int_1^3 \frac{d\,t}{a(t)} = \int_1^2 + \int_2^3$$

$$= \int_3^4 + \int_2^3$$

$$= \int_2^4 \frac{d\,t}{a(t)}$$

Events 1 and 3 are separated by the time to emit two wave crests, λ/c. $\int_1^3 \frac{d\,t}{a(t)} = \lambda/c/a(t_1)$.

$$\frac{\lambda_1}{a_1} = \frac{\lambda_2}{a_2}$$

Stated in words:

The wavelength of light expands by the same factor as the universe.

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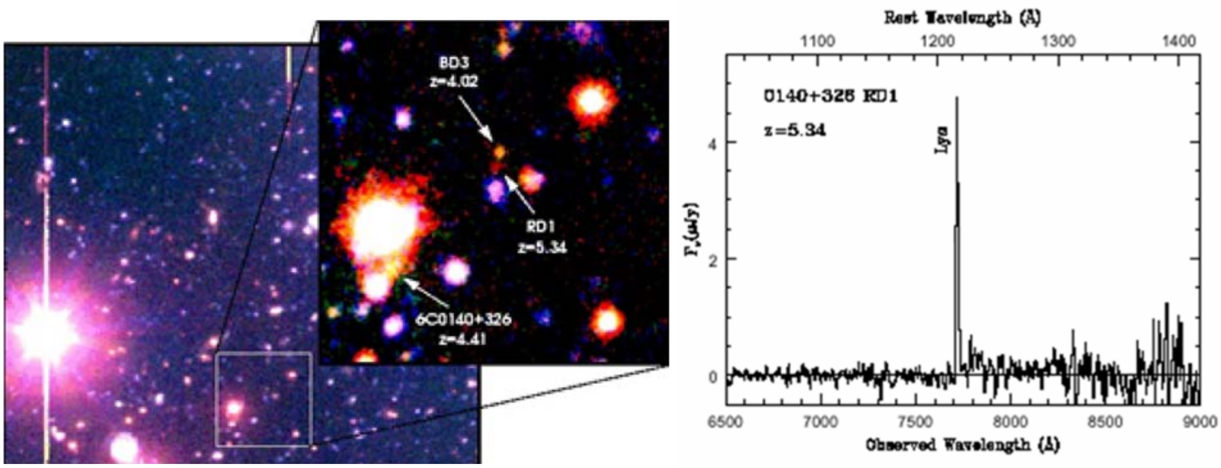
```
fig[] := Module[{n = 3, e}, e = 2 n π;
   Plot[{Sin[x], -Sin[x]}, {x, -π / 6, e + π / 6}, Epilog → {PointSize -> Large, Point[{0, 0}],
      Point[{e, 0}], Text["MW", {0, 0}, {-1.5, 0}], Text["DG", {e, 0}, {1.5, 0}]},
     ImageSize → 200, Axes → False, BaseStyle → {FontFamily → "Helvetica", FontSize → Medium}]];
```

Example



When the light that we see left Galaxy 0140+326 RD1, its wavelength was 121.5 nm. We see its wavelength to be 771.0nm. By what factor has the universe gotten bigger?

Redshift derived from isotropy

```
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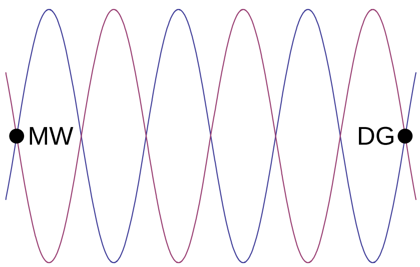
```



Space is permeated with standing waves of light. (To get a standing wave, waves goes in opposite directions.)

At some time 1, the Milky Way and a distant galaxy are N waves apart. (Here $N = 3$.) Later, MW and DG have moved apart by the factor a_2/a_1. Does the MW move to the right or the left of the node that it was on? Since there are no special directions, the MW has to stay on the node. Same for DG. Therefore

$$D_1 = N\,\lambda_1$$

$$D_2 = N\,\lambda_2$$

$$\frac{a_2}{a_1} = \frac{D_2}{D_1} = \frac{\lambda_2}{\lambda_1}$$

The wavelength of light expands by the same factor as does the universe.

More normal form: a_r at reception is 1. a_e at emission is written as a.

$$\lambda_r = \lambda_e / a.$$

Redshift is defined to be $z = (\lambda_r - \lambda_e)/\lambda_e$.

Then

$$z = a^{-1} - 1$$

$$a = (1 + z)^{-1}$$

```
fig[α_: π / 3] := Module[{c = {0, 0}, g = {Sin@α, Cos@α}},
  ListPlot[c, AspectRatio → Automatic, PlotRange → {1.05 {-1, 1}, 1.05 {-1, 1}},
   Epilog → {Circle[c, 1], PointSize → Medium, Point[c], Point[g],
     Text["x", (g + {0, 0}) / 2, {-2, 1}], Text["MW", c, {0, 1}], Text["Gal", g, {-1, -1}],
     Text["ρ", .4 {Sin[-α], Cos[-α]}, {0, -2}], Text["M", .8 {Sin[π], Cos[π]}, {0, -2}],
     Dashed, Line[{c, {Sin@α, Cos@α}}]}, ImageSize → 200,
   Axes → False, BaseStyle → {FontFamily → "Helvetica", FontSize → Medium}]];
```

Friedman's equation

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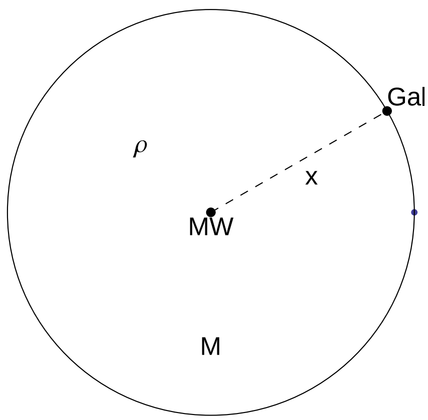
```
fig[π / 3]
```



Consider a big sphere centered on the Milky Way. There is a galaxy on the surface of the sphere. The sphere grows as the universe expands. Its distance is x, and its comoving coordinate is r.

The mass inside is slow stuff: $v \ll 1$. Galaxies are slow, since $v \sim 300\,\text{km}/\text{s} = \ 0.001$. There is a total mass M inside the sphere. The mass density is ρ.

The energy of the galaxy (of unit mass) is

$$\frac{1}{2} v^2 - \frac{G M}{x} = \text{constant}.$$

Insert Hubble's Law $v = H x$ and $H = \frac{1}{a} \frac{d a}{d t}$

Insert $M = \frac{4 \pi}{3} \rho x^3$ to get

$$\left(\frac{1}{a} \frac{d a}{d t}\right)^2 - \frac{8 \pi}{3} G \rho = \frac{\text{constant}}{r^2 a^2}$$

Since the LHS does not depend on which galaxy I pick, the RHS should be independent of r. The constant must depend on r^2.

$$\left(\frac{1}{a} \frac{d a}{d t}\right)^2 - \frac{8 \pi}{3} G \rho = \frac{\text{constant}}{a^2}$$

What is the constant? It comes from E's field equation

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Here we have to use a result that we do not have until we have Einstein's field equation. The constant is $-1/r_0^2$. It is related to the radius of curvature in the metric. This is Friedman's equation:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 - \frac{8\pi}{3} G \rho = -\frac{1}{r_0^2 a^2}$$

$$H^2 - \frac{8\pi}{3} G \rho = -\frac{1}{r_0^2 a^2}$$

Interpretation?

KE + PE = Total

The KE term became H^2. Why?

The PE term became $G\rho$. Why?

The total energy term became $-\frac{1}{r_0^2 a^2}$.

Curvature is related to mass density

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Define H_0 to be Hubble's constant at the present time, and $\rho_0$to be the mass density at the present time.
At the present time,

$$H_0^2 - \frac{8\pi}{3} G \rho_0 = -\frac{1}{r_0^2}$$

$$(r_0 H_0)^2 = \left(\frac{8\pi G \rho_0}{3 H_0^2} - 1\right)^{-1}$$

Interpretation: $H_0^{-1}$is a length, called the Hubble length. The radius of curvature r_0 compared to the Hubble length is the RHS. The RHS involves the mass density and Hubble's constant.

The quantity is called the density parameter. $\Omega_0 \equiv \frac{8\pi G \rho_0}{3 H_0^2}$.

Interpretation: Ω is the ratio of the $|\text{PE}|$ to the KE.

$$(r_0 H_0)^{-2} = \Omega_0 - 1$$

Suppose $r_0$is infinity. Then the density parameter equals the critical density $\rho_0 = \frac{3 H_0^2}{8\pi G}$.

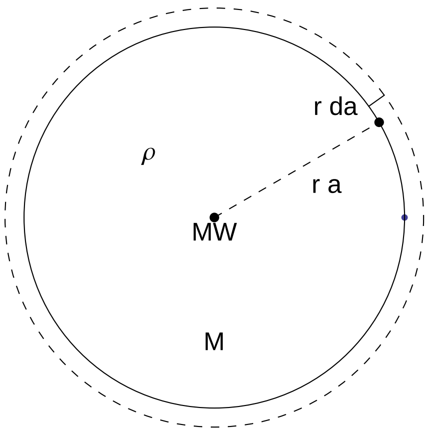
Suppose $r_0^2 > 0$. Then $\Omega_0 > 1$, and the density is greater than the critical density.
Suppose $r_0^2 < 0$. Then $\Omega_0 < 1$, and the density is less than the critical density.

In[30]:=
```
fig[α_: π / 3] := Module[{c = {0, 0}, g = {Sin@α, Cos@α}, gp, αp = α - .1},
   gp = {Sin@αp, Cos@αp};
   ListPlot[c, AspectRatio → Automatic, PlotRange → 1.12 {{-1, 1}, {-1, 1}},
    Epilog → {Circle[c, 1], PointSize → Medium, Point[c], Point[g],
      Text["r a", (g + {0, 0}) / 2, {-2, 1}], Text["r da", gp, {1.5, 0}], Text["MW", c, {0, 1}],
      Text["ρ", .4 {Sin[-α], Cos[-α]}, {0, -2}], Text["M", .8 {Sin[π], Cos[π]}, {0, -2}],
      Line[{gp, 1.1 gp}], Dashed, Line[{c, g}], Circle[c, 1.1]}, ImageSize → 200,
    Axes → False, BaseStyle → {FontFamily → "Helvetica", FontSize → Medium}]];
```

"Matter" with pressure

In[31]:= `fig[π / 3]`

Out[31]=



The energy inside the sphere changes as the universe expands because the inside does work on the outside. Energy density is ρ.

$$d E = -p A d x = -p d V$$

$$d \rho a^3 = -p d a^3.$$

Cases:

1) Pressureless matter. The speed of galaxies is 300km/s=0.001. The pressure is 10^{-6}.

$$d \left(\rho a^3\right) = 0$$

$$\rho \sim a^{-3}$$

2) Radiation has pressure.

[For a photon, $E^2 = p^2 + m^2$ becomes $E = | p |$. The pressure in one direction is $p_x^2 / E = \frac{1}{3} E$. The pressure is proportional to the momentum p_x times the rate at which photons are hitting p_x / E. Here, p is momentum.] Therefore

$$p = \frac{1}{3} \rho$$

Put that in

$$d \rho a^3 = -p d a^3$$

to get

$$\rho \sim a^{-4}$$

3) Vacuum. The vacuum has energy density from the quantum mechanical principle $\Delta E \Delta t > h$. Each field has some "vacuum energy." This energy density is a property of the field and therefore does not depend on the expansion of the universe.

$$\rho \sim \text{constant}$$

Then the pressure is negative.

$$d \rho a^3 = \rho d a^3.$$

$$\rho d a^3 = -p d a^3$$

$$p = -\rho$$

The pressure of the material inside the sphere is pulling on the outside, not pushing.

For pressureless matter

$\rho \sim a^{-3}$.

For radiation

$\rho \sim a^{-4}$.

For the vacuum

$\rho \sim \text{constant}$.