Riemann-Christoffel curvature tensor—23 Mar 2010

- Reading: Weinberg Gravitation & Cosmology, §6 and Hartle §21.3
- Outline
  - Finish covariant derivatives
  - Riemann-Christoffel curvature tensor
  - Bianchi identity

Covariant derivative of a contravariant vector

How do you take derivatives of tensors?

We already found that the equation of motion is

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma = 0.$$

The terms $\frac{du^\alpha}{d\tau}$ and $\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma$ are not tensors. Proof: $\Gamma^\alpha{}_{\beta\gamma}\, u^\beta\, u^\gamma$ is zero in a gravity-free frame. If it were a tensor, it must be zero in all frames.

We derived the equation of motion by differentiating the 4-velocity.
Rewrite

$$\frac{du^\alpha}{d\tau} = \frac{dx^\beta}{d\tau}\frac{\partial u^\alpha}{\partial x^\beta} = u^\beta \frac{\partial u^\alpha}{\partial x^\beta}$$

and insert to get

$$u^\beta\left(\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, u^\gamma\right) = 0.$$

This says: In the parenthesis is the change in u^α in the x^β direction. Contracting it (taking the dot product) with u^β results in 0.

Contraction is a tensor operation. $\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, u^\gamma$ is a tensor.
For any contravarient vector A^α,

$$\boxed{\nabla_\beta A^\alpha = \frac{\partial A^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\beta\gamma}\, A^\gamma}$$

is a tensor. This is called the covariant derivative. Another notation:

$$\boxed{A^\alpha{}_{;\beta} = A^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma}\, A^\gamma}$$

Q: Is $A^\alpha{}_{;\beta} \equiv \nabla_\beta A^\alpha$ covariant or contravarient in the index β?

xxx
Example: For 2-dimensional polar coordinates, the metric is

$$\mathrm{ds}^2 = \mathrm{dr}^2 + r^2\, \mathrm{d}\theta^2$$

The non-zero Christoffel symbols are (8.17)

$$\Gamma^r_{\theta\theta} = -r$$

$$\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = 1/r.$$

$$A^r{}_{;r} = A^r{}_{,r}$$

$A^r{}_{;\theta} = A^r{}_{,\theta} - r\, A^\theta$
$A^\theta{}_{;r} = A^\theta{}_{,r} + 1/r\, A^\theta$
$A^\theta{}_{;\theta} = A^\theta{}_{,\theta} + 1/r\, A^r$

The covariant derivative of the r component in the r direction is the regular derivative. If a vector field is constant, then $A^r{}_{;r} = 0$. The covariant derivative of the r component in the θ direction is the regular derivative plus another term. Even if a vector field is constant, $A^r{}_{;\theta} \neq 0$. The Γ term accounts for the change in the coordinates.

The idea of a covariant derivative of a vector field A in the direction a. Is this a good definition?

$\nabla_a A^\alpha = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [A(x + \epsilon\, a) - A(x)]$???

However, the components of $A(x + \epsilon\, a)$ may be different even if the vector is the same, because the coordinates are changing. We must move $A(x + \epsilon\, a)$ back to x before comparing. Moving is called parallel transporting. This is what the Γ term does.

$\nabla_a A^\alpha = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{\text{parallel transport}[A(x + \epsilon\, a)] - A(x)\}$

Q: Simplicio: Covariant derivatives are irrelavant. I want to know about gravity. In what way is Simplicio mistaken?

◄ | ►

Covariant derivative of a tensor

A contravariant tensor is $A^\mu B^\nu$. The covariant derivative of it is

$$(A^\mu B^\nu)_{;\alpha} = A^\mu (B^\nu)_{;\alpha} + (A^\mu)_{;\alpha} B^\nu$$
$$= A^\mu\left(B^\nu{}_{,\alpha} + \Gamma^\nu_{\beta\alpha} B^\beta\right) + B^\nu\left(A^\mu{}_{,\alpha} + \Gamma^\mu_{\beta\alpha} A^\beta\right)$$
$$= (A^\mu B^\nu)_{,\alpha} + A^\mu B^\beta \Gamma^\nu_{\beta\alpha} + B^\nu A^\beta \Gamma^\mu_{\beta\alpha}$$

Therefore the covariant derivative of a contravariant tensor is

$$\boxed{T^{\mu\nu}{}_{;\alpha} = T^{\mu\nu}{}_{,\alpha} + T^{\mu\beta} \Gamma^\nu_{\beta\alpha} + T^{\beta\nu} \Gamma^\mu_{\beta\alpha}}.$$

There is one Christoffel symbol for each upper index.

The covariant derivative of a covariant vector is

$$\boxed{A_{\alpha;\beta} = A_{\alpha,\beta} - \Gamma^\gamma{}_{\alpha\beta} A_\gamma}$$

Proof: Find the covariant derivative of $A_\alpha A^\alpha$.

The covariant derivative of a mixed tensor: Put in $+\Gamma$ for each upper index and $-\Gamma$ for each lower index.

◄ | ►

How to measure curvature

Q: In what object is gravity encoded? What does the Equivalence Principle say?

Q: Can you measure curvature by looking at a point?

◄ | ►

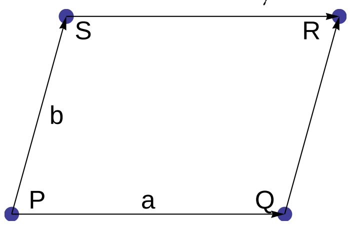
How to measure curvature

Q: In what object is gravity encoded? What does the Equivalence Principle say? Gravity is encoded in a general coordinate transformation.

Q: Can you measure curvature by looking at a point? No. The equivalence principle says that gravity can be removed in a small region of space-time by a coordinate transformation. You must explore a region that is not small.

Q: How to detect curvature of the Earth's surface.
Carry a vector, which points east, from the north pole to the equator.

Consider a vector field A_γ. Move from point P to Q to R. Move from P to S to R. Compare.



The change in A in going from P to Q is

$$\mathrm{dA}_{\gamma\,\mathrm{PQ}} = \left(\frac{\partial A_\gamma}{\partial x^\alpha}\right) a^\alpha$$

Q: Why is this not a tensor equation?

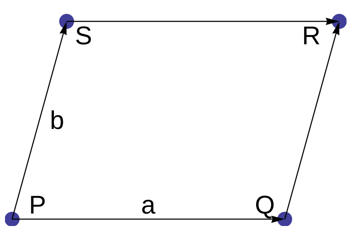
◄ | ►

In[54]:=
```
fig[] := Module[{p, q, r, s, x},
  p = {0, 0}; q = {1, 0}; s = {.2, 1}; r = {1.2, 1};
  x = {p, q, r, s};

  ListPlot[x, PlotStyle → {PointSize → Large},
   Epilog → {Text["P", p, {-3, -1}], Text["Q", q, {2, -1}], Text["S", s, {-2, 1}],
     Text["R", r, {3, 1}], Text["a", Mean@{x⟦1⟧, x⟦2⟧}, {0, -1}],
     Text["b", Mean@{x⟦1⟧, x⟦4⟧}, {-2.5, 0}], Arrow[{x⟦1⟧, x⟦2⟧}], Arrow[{x⟦2⟧, x⟦3⟧}],
     Arrow[{x⟦1⟧, x⟦4⟧}], Arrow[{x⟦4⟧, x⟦3⟧}]}, bs, Axes → None, ImageSize → 160]
 ]
```

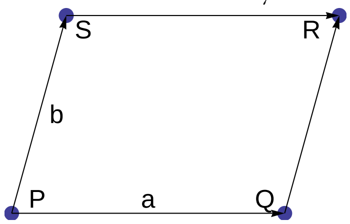
In[55]:=
```
fig[]
```

Out[55]=



How to measure curvature

Consider a vector field A_γ. Move from point P to Q to R. Move from P to S to R. Compare.



The change in A_γ in going from P to Q is

$$d\,A_{\gamma\,\mathrm{PQ}} = \left(\frac{\partial A_\gamma}{\partial x^\alpha}\right) a^\alpha$$

Q: Why is this not a tensor equation? The derivative of a vector field is not a tensor. Use the covarient derivative

$$\nabla_\alpha A_\gamma = \frac{\partial A_\gamma}{\partial x^\alpha} - \Gamma^\sigma{}_{\gamma\alpha}\, A_\sigma$$

This is a tensor equation:

$$d\,A_{\gamma\,\mathrm{PQ}} = \nabla_\alpha A_\gamma\, a^\alpha$$

The change in A in going P→Q→R is

$$d\,\mathrm{A}\gamma_{\,\mathrm{PQR}} = \nabla_\beta \left(\nabla_\alpha A_\gamma\right) a^\alpha\, b^\beta$$

The change in A in going P→S→R is

$$d\,A_{\gamma\,\mathrm{PSR}} = \nabla_\alpha \left(\nabla_\beta A_\gamma\right) a^\alpha\, b^\beta$$

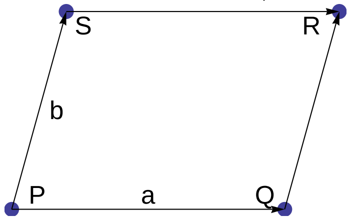
The change in a round trip P→Q→R→S→P is

$$d\,A_{\gamma\,\mathrm{PQR}} - d\,A_{\gamma\,\mathrm{PSR}} = \left[\nabla_\beta\left(\nabla_\alpha A_\gamma\right) - \nabla_\alpha\left(\nabla_\beta A_\gamma\right)\right] a^\alpha\, b^\beta$$

Q: In MA1, I learned that $\frac{\partial^2}{\partial x\,\partial y} = \frac{\partial^2}{\partial y\,\partial x}$. Why doesn't the quantity in brackets [] = 0?

How to measure curvature

Consider a vector field A_γ. Move from point P to Q to R. Move from P to S to R. Compare.



The change in A_γ in going from P to Q is

$$d\,A_{\gamma\,\text{PQ}} = \left(\frac{\partial A_\gamma}{\partial x^\alpha}\right) a^\alpha$$

Q: Why is this not a tensor equation? The derivative of a vector field is not a tensor. Use the covarient derivative

$$\nabla_\alpha A_\gamma = \frac{\partial A_\gamma}{\partial x^\alpha} - \Gamma^\sigma{}_{\gamma\alpha} A_\sigma$$

The change in a round trip P→Q→R→S→P is

$$d\,A_{\gamma\,\text{PQR}} - d\,A_{\gamma\,\text{PSR}} = \left[\nabla_\beta\left(\nabla_\alpha A_\gamma\right) - \nabla_\alpha\left(\nabla_\beta A_\gamma\right)\right] a^\alpha\, b^\beta$$

Q: In MA1, I learned that $\frac{\partial^2}{\partial x\,\partial y} = \frac{\partial^2}{\partial y\,\partial x}$. Why doesn't the quantity in brackets [] = 0? The parts involving partial derivatives of the vector A_γ is 0. The remaining parts involve the Christoffel symbol times A. Therefore, the nonzero part can be written as

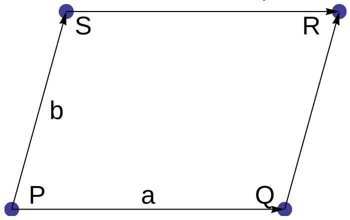
$$d\,A_{\gamma\,\text{PQR}} - d\,A_{\gamma\,\text{PSR}} = -A_\sigma\, R^\sigma{}_{\gamma\alpha\beta}\, a^\alpha\, b^\beta.$$

What does this say?

Q: In a round trip, a vector field A_γ changes by the contraction of what?

Riemann-Christoffel curvature tensor

Consider a vector field A_γ. Move from point P to Q to R. Move from P to S to R. Compare.



The change in A_γ in going from P to Q is

$$d\,A_{\gamma\,\text{PQ}} = \left(\frac{\partial A_\gamma}{\partial x^\alpha}\right) a^\alpha$$

Q: Why is this not a tensor equation? The derivative of a vector field is not a tensor. Use the covarient derivative

$$\nabla_\alpha A_\gamma = \frac{\partial A_\gamma}{\partial x^\alpha} - \Gamma^\sigma{}_{\gamma\alpha} A_\sigma$$

The change in a round trip P→Q→R→S→P is

$$d\,A_{\gamma\,\text{PQR}} - d\,A_{\gamma\,\text{PSR}} = \left[\nabla_\beta\left(\nabla_\alpha A_\gamma\right) - \nabla_\alpha\left(\nabla_\beta A_\gamma\right)\right] a^\alpha\, b^\beta$$

Q: In MA1, I learned that $\frac{\partial^2}{\partial x\,\partial y} = \frac{\partial^2}{\partial y\,\partial x}$. Why doesn't the quantity in brackets [] = 0? The parts involving partial derivatives of the vector A_γ is 0. The remaining parts involve the Christoffel symbol times A. Therefore, the nonzero part can be written as

$$\boxed{d\,A_{\gamma\,\text{PQR}} - d\,A_{\gamma\,\text{PSR}} = -A_\sigma\, R^\sigma{}_{\gamma\alpha\beta}\, a^\alpha\, b^\beta}$$

What does this say?

Q: In a round trip, a vector field A_γ changes by the contraction of A, a tensor R, the position change a, and the position change b.

The tensor $R^\sigma{}_{\gamma\alpha\beta}$is called the Riemann-Cristoffel curvature tensor.

Q: If I swap α and β, is R the same? $R^\sigma{}_{\gamma\alpha\beta} = R^\sigma{}_{\gamma\beta\alpha}$? What are the last two indices for?

Calculating $R^\sigma{}_{\gamma\alpha\beta}$:

$$\nabla_\beta\left(\nabla_\alpha A_\gamma\right) = \nabla_\beta\left(\frac{\partial A_\gamma}{\partial x^\alpha} - \Gamma^\sigma{}_{\gamma\alpha} A_\sigma\right)$$

$$= \frac{\partial^2 A_\gamma}{\partial x^\beta\,\partial x^\alpha} - A_\sigma \frac{\partial}{\partial x^\beta}\Gamma^\sigma{}_{\gamma\alpha} - \Gamma^\sigma{}_{\gamma\alpha}\frac{\partial}{\partial x^\beta}A_\sigma - \Gamma^\sigma{}_{\gamma\beta}\left(\frac{\partial A_\gamma}{\partial x^\alpha}\, A_\sigma\right)$$

We can ignore the partial derivatives of A, because in the end only the terms in A survive.

It is possible to show that

$$R^\sigma{}_{\gamma\alpha\beta} = \frac{\partial}{\partial x^\alpha}\Gamma^\sigma{}_{\gamma\beta} - \frac{\partial}{\partial x^\beta}\Gamma^\sigma{}_{\gamma\alpha} + \Gamma^\sigma{}_{\alpha\epsilon}\Gamma^\epsilon{}_{\gamma\beta} - \Gamma^\sigma{}_{\beta\epsilon}\Gamma^\epsilon{}_{\gamma\alpha}$$

Ricci tensor and curvature scalar, symmetry

The Ricci tensor is a contraction of the Riemann-Christoffel tensor

$$R_{\gamma\beta} \equiv R^{\alpha}{}_{\gamma\alpha\beta}.$$

The curvature scalar is the contraction of the Ricci tensor

$$R = g^{\beta\gamma} R_{\gamma\beta}.$$

Symmetry properties of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta} \equiv g_{\alpha\sigma} R^{\sigma}{}_{\beta\gamma\delta}$

1) Symmetry is swapping the first and second pair

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

2) Antisymmetry in swapping first pair or second pair

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$

3) Cyclicity in the last three indices.

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$$

◄ | ►

Example: Surface of a 2-d sphere

The metric is

$$\mathrm{ds}^2 = a^2\left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right).$$

The nonzero parts of the Christoffel symbol are

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = -\sin\theta\cos\theta$$

The Riemann-Christoffel tensor is in general

$$R^{\sigma}{}_{\gamma\alpha\beta} = \frac{\partial}{\partial x^{\alpha}}\Gamma^{\sigma}{}_{\gamma\beta} - \frac{\partial}{\partial x^{\beta}}\Gamma^{\sigma}{}_{\gamma\alpha} + \Gamma^{\sigma}{}_{\alpha\epsilon}\Gamma^{\epsilon}{}_{\gamma\beta} - \Gamma^{\sigma}{}_{\beta\epsilon}\Gamma^{\epsilon}{}_{\gamma\alpha}$$

Q: Compute one non-zero component (no sum)

$$R^{\theta}{}_{\phi\theta\phi} = \ldots = \sin^2\theta$$

Q: Compute (no sum)

$$R^{\theta\phi}{}_{\theta\phi}$$

Q: Compute the Ricci tensor. Answer:

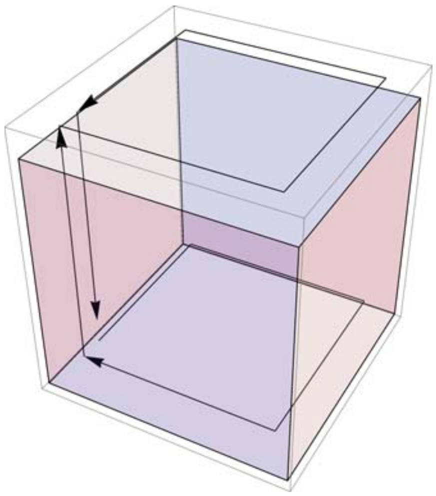
$$R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi} = a^{-2}$$

$$R^{\theta}{}_{\phi} = R^{\phi}{}_{\theta} = 0$$

Q: Compute the curvature scalar R

Bianchi identity

Bianchi's identity: The curvature induced change of a vector carried over the 6 faces of a cube is zero.
Proof: Each side is traversed twice (or 4 times) in opposite directions.



In equation form:
The change on the y-z at face at x is

$$d\,A_\sigma(x) = -A_\sigma\, R^\sigma{}_{\gamma\,y\,z}(x)\,\text{dy dz}$$

The change on the y-z face at $x + \text{dx}$ is

$$d\,A_\sigma(x + d\,x) = -A_\sigma\, R^\sigma{}_{\gamma\,y\,z}(x + \text{dx})\,\text{dy dz}$$

The change over both faces is

$$d\,A_\sigma(x + d\,x) - d\,A_\sigma(x) = -A_\sigma\, \nabla_x R^\sigma{}_{\gamma\,y\,z}\,\text{dx dy dz}$$

Q: Is $\nabla_x R^\sigma{}_{\gamma\,y\,z}$ the same as $\frac{\partial}{\partial x} R^\sigma{}_{\gamma\,y\,z}$?

Traverse the face at $x + \text{dx}$ in the outward-pointing sense and the face at x in the outward-pointing sense.

The change over all 6 faces is

$$A_\sigma\,\text{dx dy dz}\left(\nabla_x R^\sigma{}_{\gamma\,y\,z} + \nabla_y R^\sigma{}_{\gamma\,z\,x} + \nabla_z R^\sigma{}_{\gamma\,x\,y}\right)$$

and since each side in traversed in opposite directions, it is zero.

We chose x, y, and z, but we could have also chosen t for one of the directions. Therefore, we have proved the Bianchi identity,

$$\boxed{\nabla_\alpha R^\sigma{}_{\tau\beta\gamma} + \nabla_\beta R^\sigma{}_{\tau\gamma\alpha} + \nabla_\gamma R^\sigma{}_{\tau\alpha\beta} = 0}$$

A contracted form of the Bianchi identity is:

$$\boxed{\nabla_\mu\left(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu}\, R\right) = 0}$$

In[103]:=
```
fig[] := Module[{}, Show[Graphics3D[{{Opacity[.2], Cuboid[]},
      Arrow[{{.1, .1, 1.1}, {.9, .1, 1.1}, {.9, .9, 1.1}, {.1, .9, 1.1}, {.1, .2, 1.1}}],
      Arrow[Reverse@{{.1, .1, .1}, {.9, .1, .1}, {.9, .9, .1}, {.1, .9, .1}, {.1, .2, .1}}],
      Arrow[{{.1, .1, .1}, {.1, .1, 1.1}}],
      Arrow[{{.1, .2, 1.1}, {.1, .2, .2}}]}], ImageSize → 200]]
```

In[104]:=
```
fig[]
```

Out[104]=